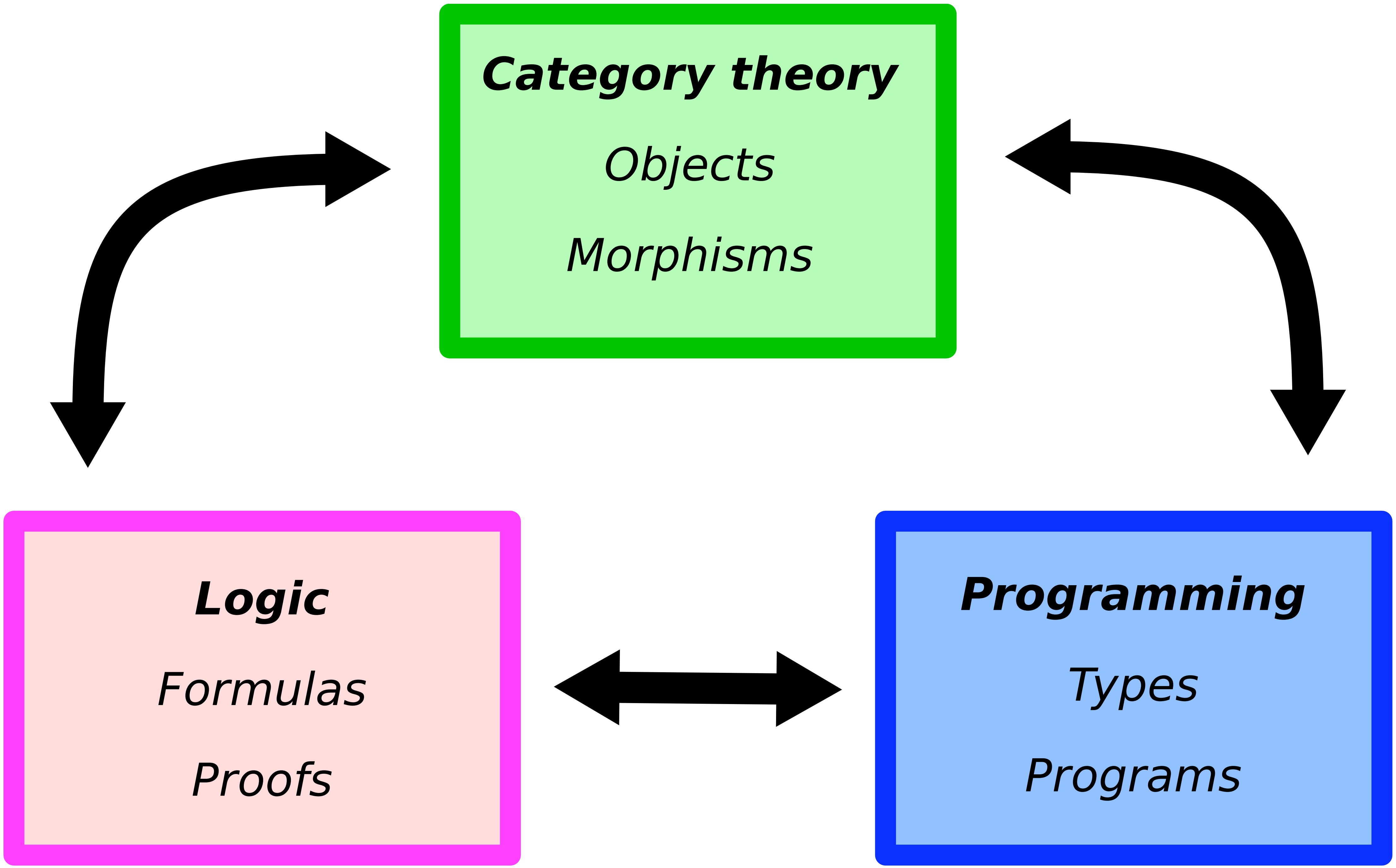
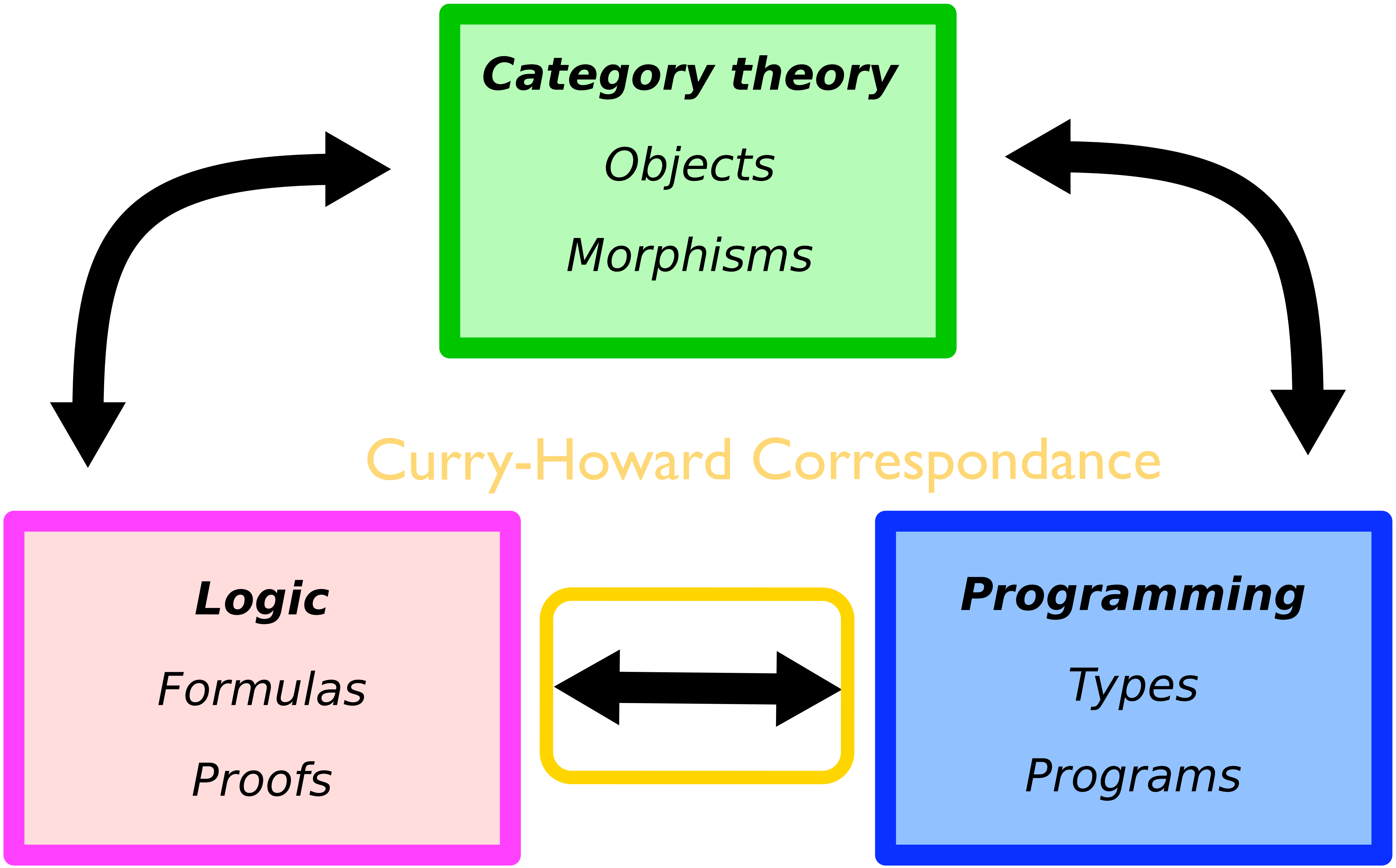
Category Theory for computer scientists

Nicolas Tabareau
INRIA, Nantes

The denotational semantics trinity



The denotational semantics trinity



The simply typed λ-calculus

variable

$$\frac{}{x : A \vdash x : A}$$

abstraction

$$\frac{\Gamma, x : A \vdash P : B}{\Gamma \vdash \lambda x.P : A \Rightarrow B}$$

application

$$\frac{\Gamma \vdash P : A \Rightarrow B \qquad \Delta \vdash Q : A}{\Gamma, \Delta \vdash PQ : B}$$

weakening

$$\frac{\Gamma \vdash P : B}{\Gamma, x : A \vdash P : B}$$

contraction

$$\frac{\Gamma, x : A, y : A \vdash P : B}{\Gamma, z : A \vdash P[x, y \leftarrow z] : B}$$

exchange

$$\frac{\Gamma, x : A, y : B, \Delta \vdash P : C}{\Gamma, y : B, x : A, \Delta \vdash P : C}$$

Intuitionistic minimal logic

axiom	$$\frac{}{A \vdash A}$$
$\Rightarrow$ I	$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$$
$\Rightarrow$ E	$$\frac{\Gamma \vdash A \Rightarrow B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B}$$
weakening	$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}$$
contraction	$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}$$
exchange	$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$$

Intuitionistic minimal logic

axiom

$$\frac{}{A \vdash A}$$

$\Rightarrow$ I

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$$

$\Rightarrow$ E

$$\frac{\Gamma \vdash A \Rightarrow B \qquad \Delta \vdash A}{\Gamma, \Delta \vdash B}$$

weakening

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}$$

contraction

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}$$

exchange

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$$

Curry
Howard

Other correspondances

Cut elimination $\Leftrightarrow$ β-reduction

Other correspondances

priority to right-hand side
in cut-elimination

$\Leftrightarrow$

call-by-name

Other correspondances

priority to <u>left-hand side</u>
in cut-elimination

$\Leftrightarrow$

call-by-value

Other correspondances

Pierce law

$$((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$$

$$\Leftrightarrow$$

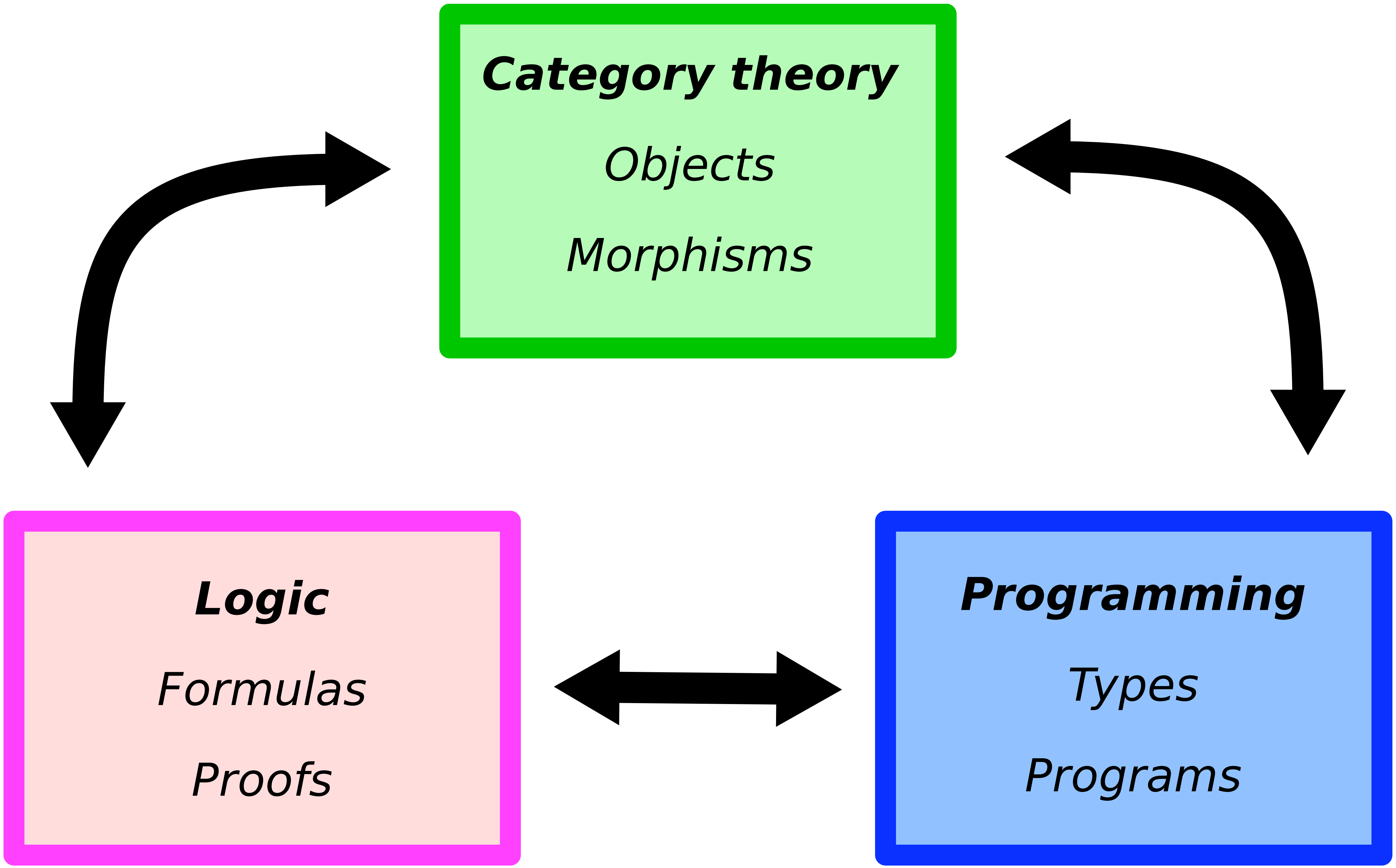
call-cc (continuation)

Other correspondances

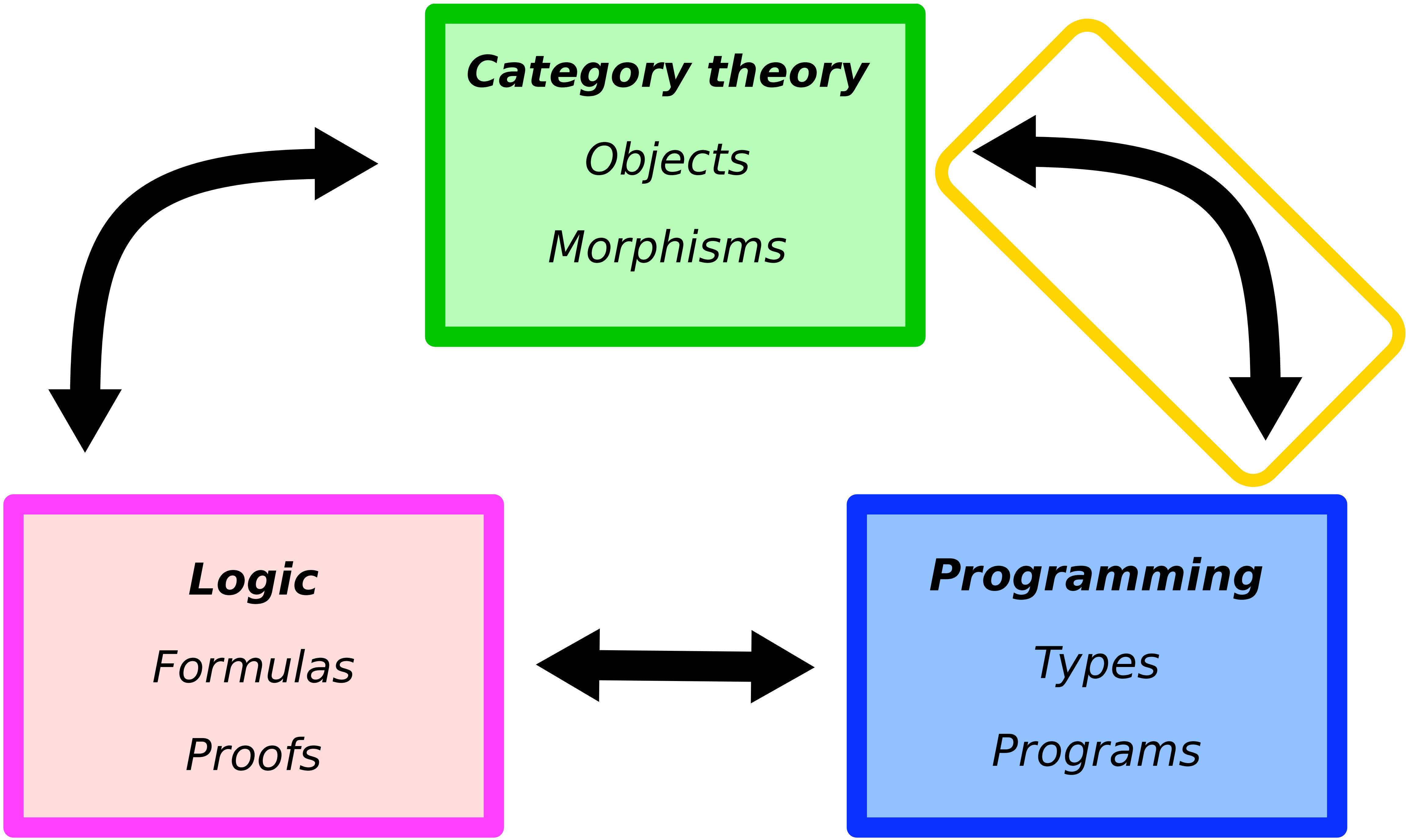
double negation translation
(Gödel translation)
$\Leftrightarrow$

continuation passing style
translation

The denotational semantics trinity



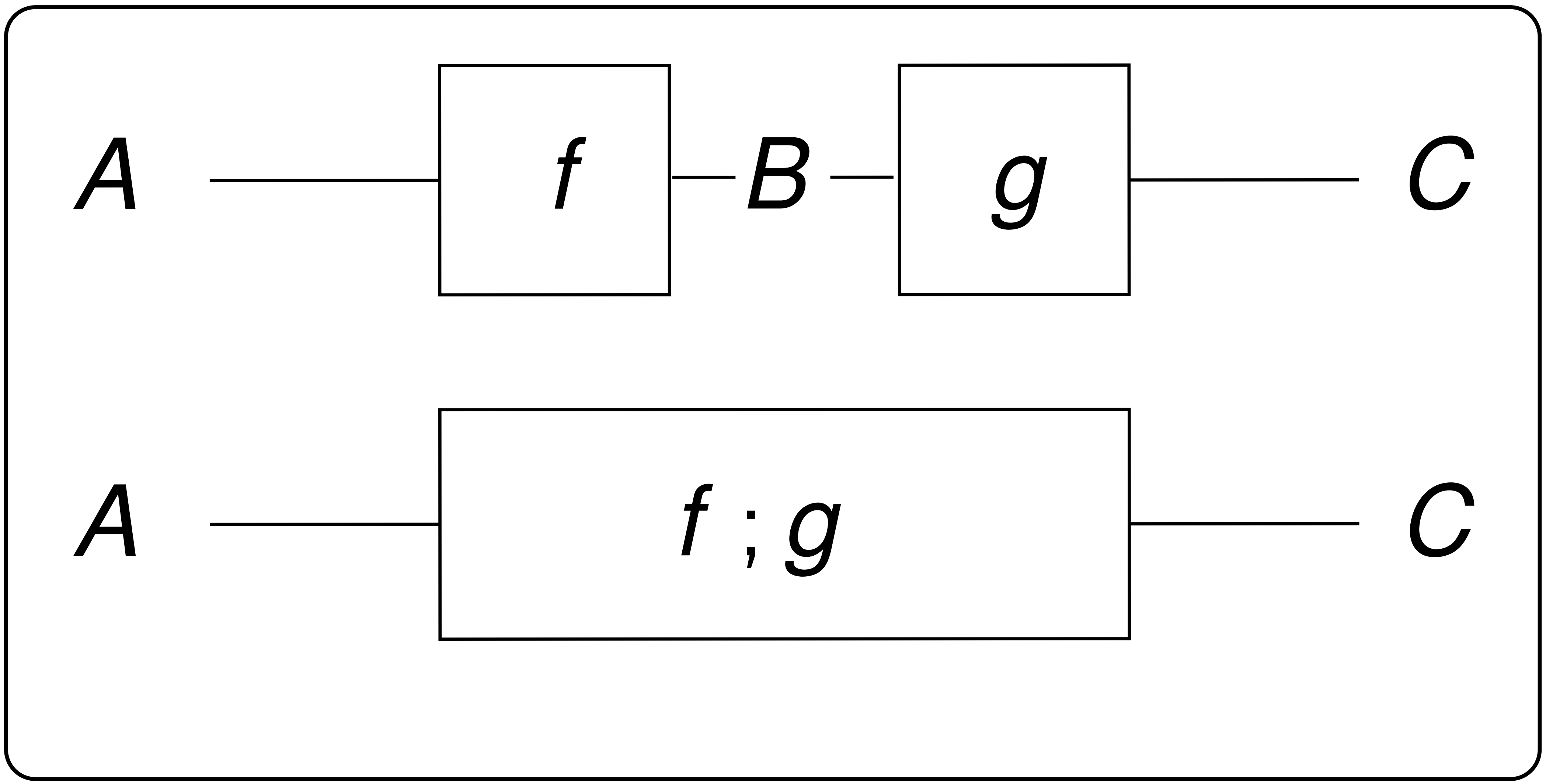
The denotational semantics trinity



What is a category ?

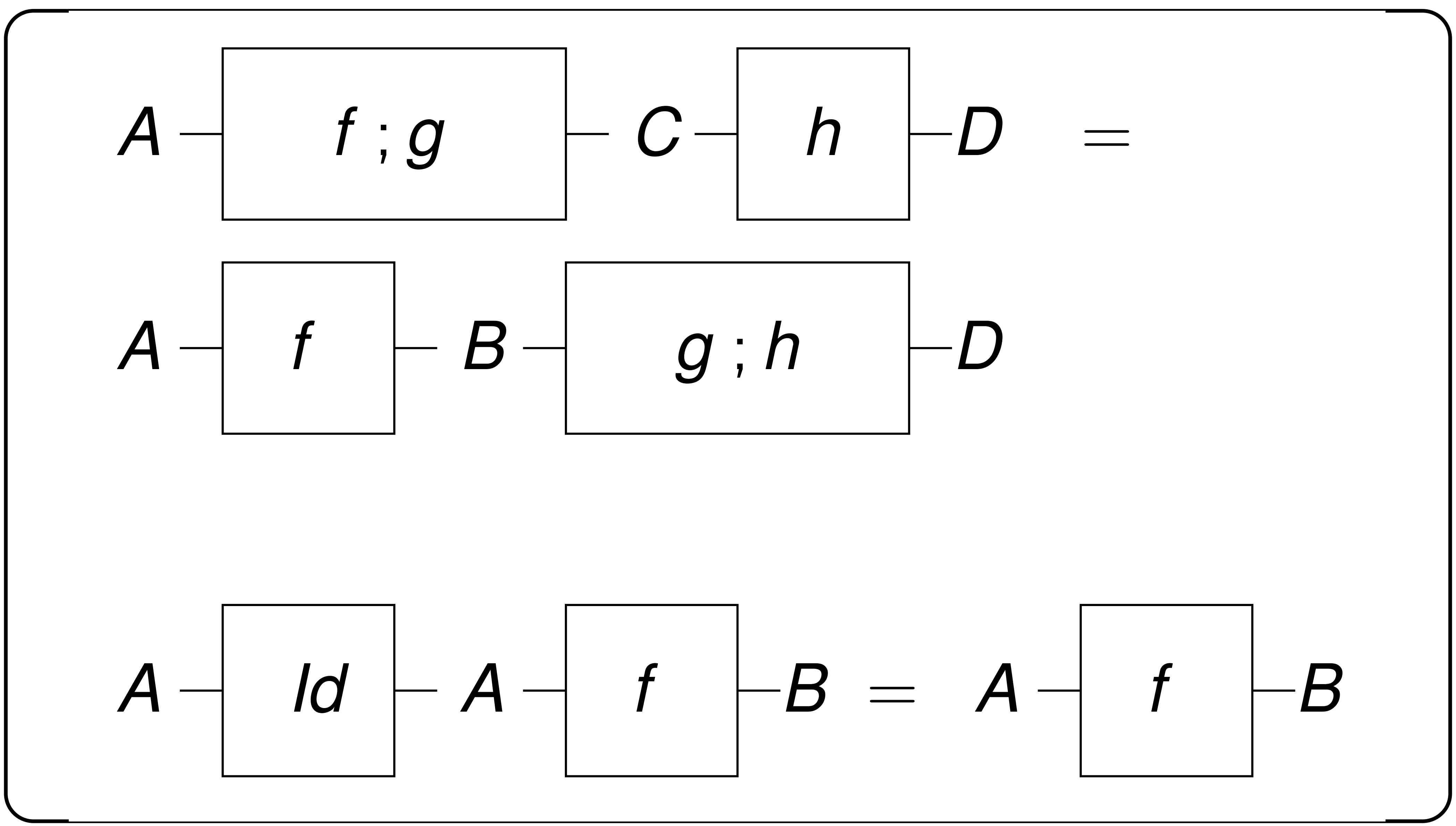
What is a category ?

Coarsely, a labelled graph
whose arrows can be composed



What is a category ?

together with basic associativity and identity rules



A first example :

A category with at most one arrow
between two objects is

A first example :

A category with at most one arrow between two objects is

A preorder

A second example :

A category with exactly
one object is

A second example :

A category with exactly
one object is

A monoid

Lo inevitable

	Objects	Morphisms
Set	sets	functions
Bij	sets	one-to-one function
Vec	vector spaces	linear applications
Ab	abelian groups	group morphisms
PO	part. order sets	monotonic functions
Dom	Scott domains	continuous function

There are also morphisms between categories

Functors

There are also morphisms between categories

Functors

Relates two categories in a structure-preserving way

Example 1:

$$U : \mathbf{Mon} \rightarrow \mathbf{Set}$$

The forgetful functor from the category of
monoids
to the category of sets.

Example 2:

U' : **Ab** → **Set**

The forgetful functor from the category of
abelian groups
to the category of sets.

Why are categories useful ?

1. Rephrase many structures with few concepts

Why are categories useful ?

2. Export abstract theorems

to concrete structures

First concept: Adjunction

$$F: \mathbf{A} \to \mathbf{B} \text{ and } G: \mathbf{B} \to \mathbf{A}$$

$$\frac{Fx \to y \text{ in } \mathbf{B}}{x \to Gy \text{ in } \mathbf{A}}$$

as many morphisms
in a natural way

First concept: Adjunction

$\mathtt{F}: \mathbf{A} \to \mathbf{B}$ and $\mathtt{G}: \mathbf{B} \to \mathbf{A}$

In that case, we say that
F is left adjoint to G

Example 1:

$U : \mathbf{Mon} \to \mathbf{Set}$

as a left adjoint

what is it?

hint : it describes a canonical way to form a monoid from a set

Example 1:

$U : \mathbf{Mon} \to \mathbf{Set}$

Answer: the word construction
(or free monoid)

Example 1:

Proof:

$$\frac{A^* \to B \text{ in } \mathbf{Mon}}{A \to U(B) \text{ in } \mathbf{Set}}$$

Take $f: A \to B$,
construct the function

$$f^*(w_1 \ldots w_n) = f(w_1) \ldots f(w_n)$$

Example 2:

$U' : \mathbf{Ab} \rightarrow \mathbf{Set}$

The left adjoint constructs
the free abelian group

Back to the point

What is the categorical structure
of the λ-calculus ?

1. we need to interpret the “,” in the typing judgment

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in the typing judgment

This is given by the notion
of product $\times$

2. we need to interpret the empty environment in the typing judgment

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This is given by the notion of terminal object

3. we need to interpret the abstraction rule

$$\frac{\Gamma, x : A \vdash P : B}{\Gamma \vdash \lambda x.P : A \Rightarrow B}$$

3. we need to interpret the abstraction rule

$$\frac{\Gamma \times A \to B}{\Gamma \to (A \Rightarrow B)}$$

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$$\frac{\Gamma \times A \to B}{\Gamma \to (A \Rightarrow B)}$$

This says that $(A \Rightarrow -)$ is

the right adjoint to $(- \times A)$

3. we need to interpret
the abstraction rule

In category terminology,
the right adjoint to the cartesian product is called the
closure

Cartesian closed category

A category with

1. a product x
2. a terminal object
3. a closure ⇒

is a cartesian closed category (CCC)

λ-calculus and CCC

We can interpret the λ-calculus in any CCC.

The interpretation is correct:

if M and N are β-equivalent
then [M] = [N]

λ-calculus and CCC

identity	$$\frac{}{x : A \vdash x : A}$$
closure (adjunction)	$$\frac{\Gamma, x : A \vdash P : B}{\Gamma \vdash \lambda x.P : A \Rightarrow B}$$
composition	$$\frac{\Gamma \vdash P : A \Rightarrow B \qquad \Delta \vdash Q : A}{\Gamma, \Delta \vdash PQ : B}$$
projection	$$\frac{\Gamma \vdash P : B}{\Gamma, x : A \vdash P : B}$$
diagonal of the product	$$\frac{\Gamma, x : A, y : A \vdash P : B}{\Gamma, z : A \vdash P[x, y \leftarrow z] : B}$$
commutativity	$$\frac{\Gamma, x : A, y : B, \Delta \vdash P : C}{\Gamma, y : B, x : A, \Delta \vdash P : C}$$

Why introducing CCC ?

Of course, for sets and functions, we don't really need category theory.

Why introducing CCC ?

But it is sometimes difficult to say that an interpretation gives rise to a model.

Why introducing CCC ?

1. Scott domains and continuous functions
2. Berry domains and stable functions
3. concrete data structures and sequential algorithms
4. opponent starting games and sequential strategies
5. ...

To be continued ...