Gradual System F: Auxiliary Definitions and Proofs

ELIZABETH LABRADA, University of Chile, Chile
MATÍAS TORO, University of Chile, Chile
ÉRIC TANTER, University of Chile, Chile

Bringing the benefits of gradual typing to a language with parametric polymorphism like System F, while preserving relational parametricity, has proven extremely challenging: first attempts were formulated a decade ago, and several designs have been recently proposed, with varying syntax, behavior, and properties. Starting from a detailed review of the challenges and tensions that affect the design of gradual parametric languages, this work presents an extensive account of the semantics and metatheory of GSF, a gradual counterpart of System F. In doing so, we also report on the extent to which the Abstracting Gradual Typing methodology can help us derive such a language. Among gradual parametric languages that follow the syntax of System F, GSF achieves a unique combination of properties. We clearly establish the benefits and limitations of the language, and discuss several extensions of GSF towards a practical programming language.

CCS Concepts: • **Theory of computation** → **Type structures**; **Program semantics**;

Additional Key Words and Phrases: Gradual typing, polymorphism, parametricity

*This work is partially funded by CONICYT/FONDECYT Regular/1190058, CONICYT/Doctorado Nacional/2015-21150510 & 21151566.

Authors' addresses: Elizabeth Labrada, University of Chile, PLEIAD Lab, Computer Science Department (DCC), Beauchef 851, Santiago, Chile; Matías Toro, University of Chile, PLEIAD Lab, Computer Science Department (DCC), Beauchef 851, Santiago, Chile; Éric Tanter, University of Chile, PLEIAD Lab, Computer Science Department (DCC), Beauchef 851, Santiago, Chile.

Contents

Contents		2
1	SF: Well-formedness	3
2	GSF: Statics	4
2.1	Syntax and Syntactic Meaning of Gradual Types	4
2.2	Lifting the Static Semantics	5
2.3	Well-formedness	8
2.4	Static Properties	8
2.4.1	Static Equivalence for Static Terms	8
2.4.2	Static Gradual Guarantee	9
3	GSF: Dynamics	14
3.1	Evidence Type Precision	14
3.2	Initial Evidence	14
3.3	Consistent Transitivity	15
3.4	GSFε: Dynamic Semantics	15
3.5	Translation from GSF to GSFε	16
4	GSF: Properties	18
4.1	Type Safety	18
4.2	Static Terms Do Not Fail	22
5	GSF and The Dynamic Gradual Guarantee	25
5.1	Evidence Type Precision	25
5.2	Monotonicity of Evidence Transitivity and Instantiation	25
5.3	Weak Dynamic Gradual Guarantee for GSF	33
5.4	Syntactic Strict Precision for GSF	41
6	GSF: Parametricity	47
6.1	Auxiliary Definitions	47
6.2	Fundamental Property	47
6.3	Contextual Equivalence	81
7	Parametricity vs. the DGG in GSF	83
8	A Cheap Theorem in GSF	84
9	Embedding Dynamic Sealing in GSF	86
10	Gradual Existential Types in GSF	110
10.1	Existential types: primitive or encoded?	110
10.2	Translation from GSF$^{\exists}$ to GSF$^{\exists}_{\varepsilon}$	111
10.3	Properties of GSF$^{\exists}$	111
10.4	GSF$^{\exists}$: Parametricity	112
10.5	A Weak Dynamic Gradual Guarantee for GSF$^{\exists}$	125

1 SF: WELL-FORMEDNESS

In this section we present auxiliary definitions for well-formedness of type name stores, and well-formedness of types.

Definition 1.1 (Well-formedness of the type name store).

$$\frac{}{\vdash \cdot} \qquad \frac{\alpha \notin \Sigma \quad \Sigma; \cdot \vdash T \quad \vdash \Sigma}{\vdash \Sigma, \alpha : T}$$

Definition 1.2 (Well-formedness of types).

$$\frac{\vdash \Sigma}{\Sigma; \Delta \vdash B} \qquad \frac{\Sigma; \Delta \vdash T_1 \quad \Sigma; \Delta \vdash T_2}{\Sigma; \Delta \vdash T_1 \rightarrow T_2} \qquad \frac{\Sigma; \Delta, X \vdash T}{\Sigma; \Delta \vdash \forall X.T} \qquad \frac{\Sigma; \Delta \vdash T_1 \quad \Sigma; \Delta \vdash T_2}{\Sigma; \Delta \vdash T_1 \times T_2}$$

$$\frac{\vdash \Sigma \quad X \in \Delta}{\Sigma; \Delta \vdash X} \qquad \frac{\vdash \Sigma \quad \alpha : T \in \Sigma}{\Sigma; \Delta \vdash \alpha}$$

2 GSF: STATICS

In this section we present auxiliary definitions and proofs of the statics semantics of GSF not presented in the paper.

2.1 Syntax and Syntactic Meaning of Gradual Types

Proposition 6.2 (Precision, inductively). *The inductive definition of type precision given in Figure 3 is equivalent to Definition 6.1.*

Proof. Direct by induction on the type structure of G_1 and G_2. We only present representative cases to illustrate the reasoning used in the proof. We prove first that $C(G_1) \subseteq C(G_2) \Rightarrow G_1 \sqsubseteq G_2$, where $G_1 \sqsubseteq G_2$ stands for the inductive definition given in Figure 3.

Case ($G_1 = B, G_2 = B$). Then $\{ B \} \subseteq \{ B \}$, but we already know that $B \sqsubseteq B$ and the result holds.

Case ($G_1 = G, G_2 = ?$). Then $C(G) \subseteq C(?) = \text{Type}$, but $G \sqsubseteq ?$ is an axiom and the result holds.

Case ($G_1 = \forall X.G_1', G_2 = \forall X.G_2'$). Then we know that $\{ \forall X.T \mid T \in C(G_1') \} \subseteq \{ \forall X.T \mid T \in C(G_1') \}$, then it must be the case that $C(G_1') \subseteq C(G_2')$. Then by induction hypothesis $G_1 \sqsubseteq G_2$, then by inductive definition of precision for type abstractions, $\forall X.G_1 \sqsubseteq \forall X.G_2$ and the result holds.

Then we prove the other direction, i.e. $G_1 \sqsubseteq G_2 \Rightarrow C(G_1) \subseteq C(G_2)$.

Case ($G_1 = B, G_2 = B$). Then $B \sqsubseteq B$, but we already know that $\{ B \} \subseteq \{ B \}$ and the result holds.

Case ($G_1 = G, G_2 = ?$). Then $G \sqsubseteq ?$, but $C(G) \subseteq C(?) = \text{Type}$ and the result holds.

Case ($G_1 = \forall X.G_1', G_2 = \forall X.G_2'$). Then we know that $\forall X.G_1 \sqsubseteq \forall X.G_2$, then by looking at the premise of the corresponding definition, $G_1' \sqsubseteq G_2'$. Then by induction hypothesis $C(G_1') \subseteq C(G_2')$. But we have to prove that $\{ \forall X.T \mid T \in C(G_1') \} \subseteq \{ \forall X.T \mid T \in C(G_1') \}$, which is direct from $C(G_1') \subseteq C(G_2')$.

□

Proposition 6.3 (Galois connection). *$\langle C, A\rangle$ is a Galois connection,* i.e.*:*
a) (Soundness) for any non-empty set of static types $S = \{ \overline{T} \}$, we have $S \subseteq C(A(S))$
b) (Optimality) for any gradual type G, we have $A(C(G)) \sqsubseteq G$.

Proof. We first proceed to prove a) by induction on the structure of the non-empty set S.

Case ($\{ B \}$). Then $A(\{ B \}) = B$. But $C(B) = \{ B \}$ and the result holds.

Case ($\{ \overline{T_{i1} \to T_{i2}} \}$). Then $A(\{ \overline{T_{i1} \to T_{i2}} \}) = A(\{ \overline{T_{i1}} \}) \to A(\{ \overline{T_{i2}} \})$. But by definition of C, $C(A(\{ \overline{T_{i1}} \}) \to A(\{ \overline{T_{i2}} \})) = \{T_1 \to T_2 \mid T_1 \in C(A(\{ \overline{T_{i1}} \})), T_2 \in C(A(\{ \overline{T_{i2}} \}))\}$. By induction hypotheses, $\{ \overline{T_{i1}} \} \subseteq C(A(\{ \overline{T_{i1}} \}))$ and $\{ \overline{T_{i2}} \} \subseteq C(A(\{ \overline{T_{i2}} \}))$, therefore $\{ \overline{T_{i1} \to T_{i2}} \} \subseteq \{ T_1 \to T_2 \mid T_1 \in \{ \overline{T_{i1}} \}, T_2 \in \{ \overline{T_{i2}} \} \} \subseteq \{T_1 \to T_2 \mid T_1 \in C(A(\{ \overline{T_{i1}} \})), T_2 \in C(A(\{ \overline{T_{i2}} \}))\}$ and the result holds.

Case ($\{ \overline{T_{i1} \times T_{i2}} \}$). We proceed analogous to case $\{ \overline{T_{i1} \to T_{i2}} \}$.

Case ($\{ X \}, \{ \alpha \}$). We proceed analogous to case $\{ B \}$.

Case ($\{ \overline{\forall X.T_i} \}$). Then $A(\{ \overline{\forall X.T_i} \}) = \forall X.A(\{ \overline{T_i} \})$. But by definition of C, $C(\forall X.A(\{ \overline{T_i} \})) = \{\forall X.T \mid T \in C(A(\{ \overline{T_i} \}))\}$. By induction hypothesis, $\{ \overline{T_i} \} \subseteq C(A(\{ \overline{T_i} \}))$, therefore $\{ \overline{\forall X.T_i} \} = \{ \forall X.T \mid T \in \{ \overline{T_i} \} \} \subseteq \{\forall X.T \mid T \in C(A(\{ \overline{T_i} \}))\}$ and the result holds.

Case ($\{ \overline{T_i} \}$ heterogeneous). Then $A(\{ \overline{T_i} \}) = ?$ and therefore $C(A(\{ \overline{T_i} \})) = \text{Type}$, but $\{ \overline{T_i} \} \subseteq \text{Type}$ and the result holds.

Now let us proceed to prove b) by induction on gradual type G.

Case (B). Trivial because $C(B) = \{ B \}$, and $A(\{ B \}) = B$.

Case ($G_1 \rightarrow G_2$). We have to prove that $A(C(G_1 \rightarrow G_2)) \sqsubseteq G_1 \rightarrow G_2$, which is equivalent to prove that $C(A(\widehat{T})) \subseteq \widehat{T}$, where $\widehat{T} = C(G_1 \rightarrow G_2) = \{ T_1 \rightarrow T_2 \mid T_1 \in C(G_1), T_2 \in C(G_2) \}$. Then $\widehat{T}$ has the form $\{ \overline{T_{i1} \rightarrow T_{i2}} \}$, such that $\forall i, T_{i1} \in C(G_1)$ and $T_{i2} \in C(G_2)$. Also note that $\{ \overline{T_{i1}} \} = C(G_1)$ and $\{ \overline{T_{i2}} \} = C(G_2)$. But by definition of A, $A(\{ \overline{T_{i1} \rightarrow T_{i2}} \}) = A(\{ \overline{T_{i1}} \}) \rightarrow A(\{ \overline{T_{i2}} \})$ and therefore $C(A(\{ \overline{T_{i1}} \}) \rightarrow A(\{ \overline{T_{i2}} \})) = \{T_1 \rightarrow T_2 \mid T_1 \in C(A(\{ \overline{T_{i1}} \})), T_2 \in C(A(\{ \overline{T_{i2}} \}))\}$. But by induction hypotheses $C(A(\{ \overline{T_{i1}} \})) \subseteq C(G_1)$ and $C(A(\{ \overline{T_{i2}} \})) \subseteq C(G_2)$ and the result holds.

Case ($G_1 \times G_2$). We proceed analogous to case $G_1 \rightarrow G_2$.

Case (X, α). We proceed analogous to case B.

Case ($\forall X.G$). We have to prove that $A(C(\forall X.G)) \sqsubseteq \forall X.G$, which is equivalent to prove that $C(A(\widehat{T})) \subseteq \widehat{T}$, where $\widehat{T} = C(\forall X.G) = \{ \forall X.T \mid T \in C(G) \}$. Then $\widehat{T}$ has the form $\{ \overline{\forall X.T_i} \}$, such that $\forall i, T_i \in C(G)$. Also note that $\{ \overline{T_i} \} = C(G)$. But by definition of A, $A(\{ \overline{\forall X.T_i} \}) = \forall X.A(\{ \overline{T_i} \})$ and therefore $C(\forall X.A(\{ \overline{T_i} \})) = \{\forall X.T \mid T \in C(A(\{ \overline{T_i} \}))\}$. But by induction hypothesis $C(A(\{ \overline{T_i} \})) \subseteq C(G)$ and the result holds.

Case (?). Then we have to prove that $C(A(?)) \subseteq C(?) = \text{TYPE}$, but this is always true and the result holds immediately.

$\square$

2.2 Lifting the Static Semantics

Definition 2.1 (Store precision). $\Xi_1 \sqsubseteq \Xi_2$ if and only if $dom(\Xi_1) = dom(\Xi_2)$ and $\forall \alpha \in dom(\Xi_1), \Xi_1(\alpha) \sqsubseteq \Xi_2(\alpha)$.

LEMMA 2.2. *If* $\Xi_1 \sqsubseteq \Xi_2, \vdash \Xi_i, G_1 \sqsubseteq G_2$, *and* $\Xi_1; \Delta \vdash G_1$, *then* $\Xi_2; \Delta \vdash G_2$.

PROOF. Straightforward induction on relation $G_1 \sqsubseteq G_2$. We only present interesting cases.

Case ($G_1 = \forall X.G_1', G_2 = \forall X.G_2'$). By definition of precision $G_1' \sqsubseteq G_1'$. By definition of well-formedness of types, $\Xi_1; X \vdash G_1'$ and then by induction hypothesis $\Xi_2; \Delta, X \vdash G_2'$. Then by definition of well-formedness of types $\Xi_2; \Delta \vdash \forall X.G_2'$ and the result holds.

Case ($G_2 = ?$). This is trivial because as $\vdash \Xi_2$, then $\Xi_2; \Delta \vdash ?$.

Case ($G_1 = \alpha, G_2 = \alpha$). Trivial by definition of $\Xi_1 \sqsubseteq \Xi_2$, $\alpha \in dom(\Xi_2)$, therefore $\alpha : G_2' \in \Xi_2$ and then $\Xi_2; \Delta \vdash \alpha$.

$\square$

LEMMA 2.3. *Let* $\Xi_1 \sqsubseteq \Xi_2$, *then* $\vdash \Xi_1 \Rightarrow \vdash \Xi_2$.

PROOF. By induction on relation $\Xi_1 \sqsubseteq \Xi_2$.

Case ($\cdot \sqsubseteq \cdot$). Trivial as $\vdash \cdot$.

Case ($\Xi_1', \alpha : G_1 \sqsubseteq \Xi_2', \alpha : G_2$). By definition of store precision we know that $\Xi_1' \sqsubseteq \Xi_2'$ and that $G_1 \sqsubseteq G_2$. By definition of well-formedness, $\vdash \Xi_1', \alpha : G_1 \Rightarrow \vdash \Xi_1'$, therefore by induction hypothesis $\vdash \Xi_2'$. We only have left to prove is that $\Xi_2'; \cdot \vdash G_2$, which follows directly from Lemma 2.2.

$\square$

Lemma 2.4. *If* $\Sigma \in C(\Xi)$ *and* $\vdash \Sigma$, *then* $\vdash \Xi$

Proof. Corollary of Lemma 2.3 as $\Sigma \sqsubseteq \Xi$. □

Lemma 2.5. *If* $\Sigma; \Delta \vdash T_1 = T_2$, *then* $\Sigma; \Delta \vdash T_1$ *and* $\Sigma; \Delta \vdash T_2$.

Proof. By induction on relation $\Sigma; \Delta \vdash T_1 = T_2$. Most cases are straightforward, so we present only the interesting cases.

Case ($T_1 = \forall X.T_1', T_2 = \forall X.T_2'$). As $\Sigma; \Delta \vdash \forall X.T_1' = \forall X.T_2'$, by inspection of the derivation rule, $\Sigma; \Delta, X \vdash T_1' = T_2'$. By induction hypotheses we know that $\Sigma; \Delta, X \vdash T_1'$, and that $\Sigma; \Delta, X \vdash T_2'$. Therefore by well-formedness of types we know that $\Sigma; \Delta \vdash \forall X.T_1'$ and that $\Sigma; \Delta \vdash \forall X.T_2'$ and the result holds.

Case ($T_1 = X, T_2 = X$). As $\Sigma; \Delta \vdash X = X$, then we know by inspection of the derivation rule that $\vdash \Sigma$ and that $X \in \Delta$. Then as $\vdash \Sigma$ and that $X \in \Delta$, $\Sigma; \Delta \vdash X$ and the result holds.

□

Proposition 6.6 (Consistency, inductively). *The inductive definition of type consistency given in Figure 3 is equivalent to Definition 6.5.*

Proof. First we prove that $\Sigma; \Delta \vdash T_1 = T_2$ for some $\Sigma \in C(\Xi)$, $T_i \in C(G_i)$ implies that $\Xi; \Delta \vdash G_1 \sim G_2$, where $\Xi; \Delta \vdash G_1 \sim G_2$ stands for the inductive definition of consistency. We proceed by straightforward induction on G_i such that the predicate holds (we only show interesting cases). By Lemma 2.4 we know that if $\vdash \Sigma$ then $\vdash \Xi$, which will be assumed to be true whenever is needed.

Case ($G_1 = B, G_2 = B$). Then $\Sigma; \Delta \vdash B = B$, but we already know that $\Xi \vdash B \sim B$ and the result holds.

Case ($G_1 = G, G_2 = ?$). We know that $\Sigma; \Delta \vdash T_1 = T_2$ for some $T_1 \in C(G)$ and $T_2 \in C(?)$. Then by Lemma 2.5, $\Sigma; \Delta \vdash T_1$, and as $\Sigma \sqsubseteq \Xi$ and $T_1 \sqsubseteq G$, by Lemma 2.2, $\Xi; \Delta \vdash G$. Then as $\Xi; \Delta \vdash G$, $G \sim ? = \text{Type}$ and the result holds.

Case ($G_1 = \forall X.G_1', G_2 = \forall X.G_2'$). Then we know that $\Sigma; \Delta \vdash \forall X.T_1 = \forall X.T_2$ where $\forall X.T_1 \in C(\forall X.G_1'), \forall X.T_2 \in C(\forall X.G_1')$. Notice that $T_1 \in C(G_1')$, $T_2 \in C(G_2')$, and that $\Sigma; \Delta, X \vdash T_1 = T_2$. Then by induction hypotheses, $\Xi \vdash G_1' \sim G_2'[\Delta, X]$, and therefore $\Xi; \Delta \vdash \forall X.G_1' \sim \forall X.G_2'$ and the result holds.

Then we prove the other direction, i.e. $G_1 \sqsubseteq G_2 \Rightarrow C(G_1) \sim C(G_2)$.

Case ($G_1 = B, G_2 = B$). Then $B \sqsubseteq B$, but we already know that $B \in C(B)$ and $\Sigma; \Delta \vdash B = B$, and the result holds immediately.

Case ($G_1 = G, G_2 = ?$). Then $G \sqsubseteq ?$. Let $T_1 \in C(G)$ and $\Sigma \in C(\Xi)$ such that $\Sigma; \Delta \vdash T_1$. As $C(?) = \text{Type}$, we can choose $T_1 \in \text{Type}$, so $\Sigma; \Delta \vdash T_1 = T_1$, and the result holds.

Case ($G_1 = \forall X.G_1', G_2 = \forall X.G_2'$). Then we know that $\Xi; \Delta \vdash \forall X.G_1' \sim \forall X.G_2'$, then by looking at the premise of the corresponding definition, $\Xi; \Delta, X \vdash G_1' \sim G_2'$. Then by induction hypotheses $\exists T_1 \in C(G_1'), T_2 \in C(G_2'), \Sigma \in C(\Xi)$, such that $\Sigma; \Delta, X \vdash T_1 = T_2$. By definition of consistency $\forall X.T_i \in C(G_i)$. Then by definition of equality, $\Sigma; \Delta \vdash \forall X.T_1 = \forall X.T_2$ and the result holds.

□

Definition 6.7 (Consistent lifting of functions). Let F_n be a function of type $\text{Type}^n \longrightarrow \text{Type}$. Its consistent lifting $F_n^\sharp$, of type $\text{GType}^n \longrightarrow \text{GType}$, is defined as: $F_n^\sharp(\overline{G}) = A(\{ F_n(\overline{T}) \mid \overline{T \in C(G)} \})$

Lemma 2.6. $G = A(C(G))$

Proof. Then we have to prove that $G = A(C(G))$. By optimality (Prop 6.3.b), we know that $A(C(G)) \sqsubseteq G$, and by soundness (Prop 6.3.a), $C(G) \subseteq C(A(C(G)))$, i.e. $G \sqsubseteq A(C(G))$. Therefore $G \sqsubseteq A(C(G))$ and $A(C(G)) \sqsubseteq G$, thus $G = A(C(G))$ and the result holds. □

Lemma 2.7. $G[G'/X] = A(\{ T[T'/X] \mid T \in C(G), T' \in C(G') \})$.

Proof. We proceed by induction on G. We only present interesting cases.

Case ($G = X$). Then $G[G'/X] = G'$, and $C(G) = \{ X \}$. Then we have to prove that $G' = A(\{ T' \mid T' \in C(G') \})$. But notice that $A(\{ T' \mid T' \in C(G') \}) = A(C(G'))$ and by Lemma 2.6 the result holds immediately.

Case ($G = ?$). Then $G[G'/X] = ?$, and $C(G) = \textsc{Type}$. Then we have to prove that $? = A(\{ T[T'/X] \mid T \in \textsc{Type}, T' \in C(G') \})$. But notice that $A(\{ T[T'/X] \mid T \in \textsc{Type}, T' \in C(G') \}) = A(C(\textsc{Type}))$ and by Lemma 2.6 the result holds immediately.

Case ($G = \forall Y.G''$). Then $G[G'/X] = \forall Y.G''[G'/X]$, and $C(G) = \forall Y.C(G'')$. Then we have to prove that $\forall Y.G''[G'/X] = A(\{ \forall Y.T''[T'/X] \mid T'' \in C(G''), T' \in C(G') \})$. But notice that by definition of abstraction $A(\{ \forall Y.T''[T'/X] \mid T'' \in C(G''), T' \in C(G') \}) = \forall Y.A(\{ T''[T'/X] \mid T'' \in C(G''), T' \in C(G') \})$. Then by induction hypothesis on G'', $G''[G'/X] = A(\{ T''[T'/X] \mid T'' \in C(G''), T' \in C(G') \})$, therefore $\forall Y.G''[G'/X] = \forall Y.A(\{ T''[T'/X] \mid T'' \in C(G''), T' \in C(G') \})$ and the result holds.

□

Proposition 6.8 (Consistent type functions). *The definitions of $dom^\sharp$, $cod^\sharp$, $inst^\sharp$, and $proj_i^\sharp$ given in Fig. 3 are consistent liftings, as per Def. 6.7, of the corresponding functions from Fig. 1.*

Proof. We present the proof for $inst^\sharp$ and $dom^\sharp$ (the other proofs are analogous).

First we prove that $inst^\sharp(G, G') = A(\widehat{inst}(C^2(G, G')))$, where $inst^\sharp(G, G')$ correspond to the algorithmic definitions presented in Fig. 3. Notice that

$$\begin{aligned}
&A(\widehat{inst}(C^2(G, G'))) \\
&= A(\widehat{inst}(\{ \langle T, T' \rangle \mid T \in C(G), T' \in C(G') \})) \\
&= A(\{ T[T'/X] \mid \forall X.T \in C(G), T' \in C(G') \})
\end{aligned}$$

But then the result follows immediately from Lemma 2.7.

Then we prove that $dom^\sharp(G) = A(\widehat{dom}(C(G)))$, where $dom^\sharp(G)$ correspond to the algorithmic definitions presented in Fig. 3. We proceed by induction on G.

Case ($G = G_1 \to G_2$). Notice that

$$\begin{aligned}
&A(\widehat{dom}(C(G))) \\
&= A(\widehat{dom}(C(G_1 \to G_2))) \\
&= A(\widehat{dom}(\{ T_1 \to T_2 \mid T_1 \in C(G_1), T_2 \in C(G_2) \})) \\
&= A(\{ T_1 \mid T_1 \in C(G_1) \}) \\
&= A(C(G_1))
\end{aligned}$$

But $dom^\sharp(G_1 \to G_2) = G_1$. Then we have to prove that $G_1 = A(C(G_1))$ which holds immediately by Lemma 2.6.

Case ($G = ?$). Notice that

$$\begin{aligned}
&A(\widehat{dom}(C(G))) \\
&= A(\widehat{dom}(C(?))) \\
&= A(\widehat{dom}(\textsc{Type})) \\
&= A(\textsc{Type}) \\
&= ?
\end{aligned}$$

and the result holds immediately as $dom^\sharp(?) = ?$.

Case ($G \neq ? \neq G_1 \to G_2$). If G has not the form $G_1 \to G_2$, or is not ?, then $dom^\sharp(G)$ is undefined. Then as $\nexists, T \in C(G)$ such that $T = T_1 \to T_2$ the result holds immediately as $dom(T)$ is undefined $\forall T \in C(G)$.

□

2.3 Well-formedness

In this section we present auxiliary definitions of the statics semantics of GSF.

Definition 2.8 (Well-formedness of type name store).

$$\frac{}{\vdash \cdot} \qquad \frac{\alpha \notin \Xi \quad \Xi; \cdot \vdash G \quad \vdash \Xi}{\vdash \Xi, \alpha : G}$$

Definition 2.9 (Well-formedness of types).

$$\frac{\vdash \Xi}{\Xi; \Delta \vdash B} \qquad \frac{\Xi; \Delta \vdash G_1 \quad \Xi; \Delta \vdash G_2}{\Xi; \Delta \vdash G_1 \to G_2} \qquad \frac{\Xi; \Delta, X \vdash G}{\Xi; \Delta \vdash \forall X.G} \qquad \frac{\Xi; \Delta \vdash G_1 \quad \Xi; \Delta \vdash G_2}{\Xi; \Delta \vdash G_1 \times G_2}$$

$$\frac{\vdash \Xi \quad X \in \Delta}{\Xi; \Delta \vdash X} \qquad \frac{\vdash \Xi \quad \alpha : G \in \Xi}{\Xi; \Delta \vdash \alpha} \qquad \frac{\vdash \Xi}{\Xi; \Delta \vdash ?}$$

2.4 Static Properties

In this section we present two static properties of GSF and the proof: the static equivalence for static terms and the static gradual guarantee.

2.4.1 *Static Equivalence for Static Terms.*

Proposition 6.9 (Static equivalence for static terms). *Let t be a static term and G a static type ($G = T$). We have $\vdash_S t : T$ if and only if $\vdash t : T$*

Proof. We prove this proposition for open terms instead. The proof is direct thanks to the equivalence between the typing rules and the equivalence between type equality and type consistency rules for static types. We only present one case to illustrate the reasoning.

First we prove $\Sigma; \Delta \vdash_S t : T \Rightarrow \Sigma; \Delta \vdash t : T$ by induction on judgment $\Sigma; \Delta \vdash_S t : T$.

Case ($\Sigma; \Delta \vdash_S t'[T''] : inst(\forall X.T', T'')$). Then $\Sigma; \Delta \vdash_S t' : \forall X.T'$, and by induction hypothesis $\Sigma; \Delta \vdash t' : \forall X.T'$. Then $inst^\sharp(\forall X.T, T'') = T[T''/X] = inst(\forall X.T', T'')$, and as $\Sigma; \Delta \vdash T''$, therefore $\Sigma; \Delta \vdash t'[T''] : T[T''/X]$ and the result holds.

Then we prove $\Sigma;\Delta \vdash t : T \Rightarrow \Sigma;\Delta \vdash_S t : T$ by induction on judgment $\Sigma;\Delta \vdash_S t : T$.

Case ($\Sigma;\Delta \vdash t'[T''] : inst^{\sharp}(\forall X.T', T'')$). Then $\Sigma;\Delta \vdash t' : \forall X.T'$, and by induction hypothesis $\Sigma;\Delta \vdash_S t' : \forall X.T'$. Then $inst(\forall X.T, T'') = T[T''/X] = inst^{\sharp}(\forall X.T', T'')$, and as $\Sigma;\Delta \vdash T''$, therefore $\Sigma;\Delta \vdash_S t'[T''] : T[T''/X]$ and the result holds.

□

2.4.2 *Static Gradual Guarantee.* In this section we present the proof of the static gradual guarantee property. In the Definition 2.10 and Definition 2.11 we present term precision and type environment precision.

Definition 2.10 (Term precision).

$$\text{(Px)}\frac{}{x \sqsubseteq x} \qquad \text{(Pb)}\frac{}{b \sqsubseteq b} \qquad \text{(P}\lambda\text{)}\frac{t \sqsubseteq t' \quad G \sqsubseteq G'}{(\lambda x : G.t) \sqsubseteq (\lambda x : G'.t')} \qquad \text{(P}\Lambda\text{)}\frac{t \sqsubseteq t'}{(\Lambda X.t) \sqsubseteq (\Lambda X.t')}$$

$$\text{(Ppair)}\frac{t_1 \sqsubseteq t_1' \quad t_2 \sqsubseteq t_2'}{\langle t_1, t_2\rangle \sqsubseteq \langle t_1', t_2'\rangle} \qquad \text{(Pasc)}\frac{t \sqsubseteq t' \quad G \sqsubseteq G'}{(t :: G) \sqsubseteq (t' :: G')} \qquad \text{(Pop)}\frac{\overline{t} \sqsubseteq \overline{t'}}{op(\overline{t}) \sqsubseteq op(\overline{t'})}$$

$$\text{(Papp)}\frac{t_1 \sqsubseteq t_1' \quad t_2 \sqsubseteq t_2'}{t_1\ t_2 \sqsubseteq t_1'\ t_2'} \qquad \text{(Papp}G\text{)}\frac{t \sqsubseteq t' \quad G \sqsubseteq G'}{t\ [G] \sqsubseteq t'\ [G']} \qquad \text{(Ppairi)}\frac{t \sqsubseteq t'}{\pi_i(t) \sqsubseteq \pi_i(t')}$$

Definition 2.11 (Type environment precision).

$$\frac{}{\cdot \sqsubseteq \cdot} \qquad\qquad \frac{\Gamma \sqsubseteq \Gamma' \quad G \sqsubseteq G'}{\Gamma, x : G \sqsubseteq \Gamma', x : G'}$$

Lemma 2.12. *If $\Xi;\Delta;\Gamma \vdash t : G$ and $\Gamma \sqsubseteq \Gamma'$, then $\Xi;\Delta;\Gamma' \vdash t : G'$ for some $G \sqsubseteq G'$.*

Proof. Simple induction on type derivation $\Xi;\Delta;\Gamma \vdash t : G$ (we only present interesting cases).

Case ($t = x$). we know that $\Sigma;\Delta;\Gamma \vdash x : G$ and $\Gamma(x) = G$. By definition of $\Gamma \sqsubseteq \Gamma'$, $\Gamma(x) \sqsubseteq \Gamma'(x)$, therefore $\Sigma;\Delta;\Gamma \vdash x : G'$, where $G \sqsubseteq G'$ and the result holds.

Case ($t = (\lambda x : G_1.t')$). we know that $\Sigma;\Delta;\Gamma \vdash (\lambda x : G_1.t') : G_1 \to G_2$, where $\Sigma;\Delta;\Gamma, x : G_1 \vdash t' : G_2$. As $\Gamma \sqsubseteq \Gamma'$ and $G_1 \sqsubseteq G_1$, then by definition of precision for type environments, $\Gamma, x : G_1 \sqsubseteq \Gamma', x : G_1'$. Therefore by induction hypothesis on $\Sigma;\Delta;\Gamma, x : G_1 \vdash t' : G_2$, $\Sigma;\Delta;\Gamma', x : G_1 \vdash t' : G_2'$, where $G_2 \sqsubseteq G_2'$. Finally, by (Gλ), $\Sigma;\Delta;\Gamma' \vdash (\lambda x : G_1.t') : G_1 \to G_2'$, and as $G_1 \to G_2 \sqsubseteq G_1 \to G_2'$, the result holds.

□

Lemma 2.13. *If $\Xi;\Delta \vdash G_1 \sim G_2$ and $G_1 \sqsubseteq G_1'$ and $G_2 \sqsubseteq G_2'$ then $\Xi;\Delta \vdash G_1' \sim G_2'$.*

Proof. By definition of $\Xi;\Delta \vdash \cdot \sim \cdot$, there exists $\langle T_1, T_2\rangle \in C^2(G_1, G_2)$ such that $T_1 = T_2$. $G_1 \sqsubseteq G_1'$ and $G_2 \sqsubseteq G_2'$ mean that $C(G_1) \subseteq C(G_1')$ and $C(G_2) \subseteq C(G_2')$, therefore $\langle T_1, T_2\rangle \in C^2(G_1', G_2')$, and the resul follows. □

Lemma 2.14. *If $G_1 \sqsubseteq G_1'$ and $G_2 \sqsubseteq G_2'$ then $G_1[G_2/X] \sqsubseteq G_1'[G_2'/X]$.*

Proof. By induction on the relation of $G_1 \sqsubseteq G_1'$. We only present interesting cases.

Case ($X \sqsubseteq X$). Then we have to prove that $X[G_2/X] \sqsubseteq X[G_2'/X]$, which is equivalent to $G_2 \sqsubseteq G_2'$, but that is part of the premise and the result holds immediately.

Case ($G_1 \sqsubseteq ?$). Then we have to prove that $G_1[G_2/X] \sqsubseteq ?$ which is always true.

Case ($\forall Y.G_3 \sqsubseteq \forall Y.G_3'$). Then we have to prove that $\forall Y.G_3[G_2/X] \sqsubseteq \forall Y.G_3'[G_2'/X]$, which is equivalent to prove that $G_3[G_2/X] \sqsubseteq G_3'[G_2'/X]$, which holds by induction hypothesis on $G_3 \sqsubseteq G_3'$.

□

Lemma 2.15. *If $G_1 \sqsubseteq G_1'$ and $G_2 \sqsubseteq G_2'$ then $inst^\sharp(G_1, G_2) \sqsubseteq inst^\sharp(G_1', G_2')$.*

Proof. By induction on relation $G_1 \sqsubseteq G_1'$.

Case ($? \sqsubseteq ?$). The result is trivial as $inst^\sharp(?, G_i') = ?$ and $? \sqsubseteq ?$.

Case ($\forall X.G_1 \sqsubseteq ?, \forall X.G_1 \sqsubseteq \forall X.G_2$). The result follows directly from Lemma 2.14.

□

Lemma 2.16. *If $G_1 \sqsubseteq G_2$ then $proj_i^\sharp(G_1) \sqsubseteq proj_i^\sharp(G_2)$.*

Proof. The proof is direct, analogous to Lemma 2.15, by induction on relation $G_1 \sqsubseteq G_2$. □

Proposition 2.17 (Static gradual guarantee for open terms). *If $\Xi; \Delta; \Gamma \vdash t_1 : G_1$ and $t_1 \sqsubseteq t_2$, then $\Xi; \Delta; \Gamma \vdash t_2 : G_2$, for some G_2 such that $G_1 \sqsubseteq G_2$.*

Proof. We prove the property on opens terms instead of closed terms: If $\Xi; \Delta; \Gamma \vdash t_1 : G_1$ and $t_1 \sqsubseteq t_2$ then $\Xi; \Delta; \Gamma \vdash t_2 : G_2$ and $G_1 \sqsubseteq G_2$.

The proof proceed by induction on the typing derivation.

Case (Gx, Gb). Trivial by definition of term precision ($\sqsubseteq$) using (Px), (Pb) respectively.

Case ($G\lambda$). Then $t_1 = (\lambda x : G_1'.t)$ and $G_1 = G_1' \mathrel{\widetilde{\rightarrow}} G_2'$. By ($G\lambda$) we know that:

$$(\mathrm{G}\lambda)\,\frac{\Xi; \Delta; \Gamma, x : G_1' \vdash t : G_2'}{\Xi; \Delta; \Gamma \vdash \lambda x : G_1'.t : G_1' \rightarrow G_2'} \tag{1}$$

Consider t_2 such that $t_1 \sqsubseteq t_2$. By definition of term precision t_2 must have the form $t_2 = (\lambda x : G_1''.t')$ and therefore

$$(\mathrm{P}\lambda)\,\frac{t \sqsubseteq t' \qquad G_1' \sqsubseteq G_1''}{(\lambda x : G_1'.t) \sqsubseteq (\lambda x : G_1''.t')} \tag{2}$$

Using induction hypotheses on the premises of (1) and (2), $\Xi; \Delta; \Gamma, x : G_1' \vdash t' : G_2''$ with $G_2' \sqsubseteq G_2''$. By Lemma 2.12, $\Xi; \Delta; \Gamma, x : G_1'' \vdash t' : G_2'''$ where $G_2'' \sqsubseteq G_2'''$. Then we can use rule ($G\lambda$) to derive:

$$(G\lambda)\,\frac{\Xi; \Delta; \Gamma, x : G_1'' \vdash t' : G_2'''}{\Xi; \Delta; \Gamma \vdash (\lambda x : G_1''.t') : G_1'' \mathrel{\widetilde{\rightarrow}} G_2'''}$$

Where $G_2 \sqsubseteq G_2''$. Using the premise of (2) and the definition of type precision we can infer that

$$G_1' \mathrel{\widetilde{\rightarrow}} G_2' \sqsubseteq G_1'' \mathrel{\widetilde{\rightarrow}} G_2'''$$

and the result holds.

Case ($G\Lambda$). Then $t_1 = (\Lambda X.t)$ and $G_1 = \forall X.G_1'$. By ($G\Lambda$) we know that:

$$(\mathrm{G}\Lambda)\,\frac{\Xi; \Delta, X; \Gamma \vdash t : G_1'}{\Xi; \Delta; \Gamma \vdash \Lambda X.t : \forall X.G_1'} \tag{3}$$

Consider t_2 such that $t_1 \sqsubseteq t_2$. By definition of term precision t_2 must have the form $t_2 = (\Lambda X.t')$ and therefore

$$(\mathrm{P}\Lambda)\,\frac{t \sqsubseteq t'}{(\Lambda X.t) \sqsubseteq (\Lambda X.t')} \tag{4}$$

Using induction hypotheses on the premises of (3) and (4), $\Xi;\Delta,X;\Gamma \vdash t' : G_1''$ with $G_1' \sqsubseteq G_1''$. Then we can use rule $(G\Lambda)$ to derive:

$$(G\Lambda)\frac{\Xi;\Delta,X;\Gamma \vdash t' : G_1''}{\Xi;\Delta;\Gamma \vdash (\lambda X.t') : \forall X.G_1''}$$

Using the definition of type precision we can infer that

$$\forall X.G_1' \sqsubseteq \forall X.G_1''$$

and the result holds.

Case (Gpair). Then $t_1 = \langle t_1', t_2' \rangle$ and $G_1 = G_1' \times G_2'$. By (Gpair) we know that:

$$(\text{Gpair})\frac{\Xi;\Delta;\Gamma \vdash t_1' : G_1' \qquad \Xi;\Delta;\Gamma \vdash t_2' : G_2'}{\Xi;\Delta;\Gamma \vdash t_1'\ t_2' : G_1' \times G_2'} \tag{5}$$

Consider t_2 such that $t_1 \sqsubseteq t_2$. By definition of term precision, t_2 must have the form $\langle t_1'', t_2'' \rangle$ and therefore

$$(\text{Ppair})\frac{t_1' \sqsubseteq t_1'' \qquad t_2' \sqsubseteq t_2''}{\langle t_1', t_2' \rangle \sqsubseteq \langle t_1'', t_2'' \rangle} \tag{6}$$

Using induction hypotheses on the premises of (5) and (6), $\Xi;\Delta;\Gamma \vdash t_1'' : G_1''$ and $\Xi;\Delta;\Gamma \vdash t_2'' : G_2''$, where $G_1' \sqsubseteq G_1''$ and $G_2' \sqsubseteq G_2''$. Then we can use rule (Gpair) to derive:

$$(\text{Gpair})\frac{\Xi;\Delta;\Gamma \vdash t_1'' : G_1'' \qquad \Xi;\Delta;\Gamma \vdash t_2'' : G_2''}{\Xi;\Delta;\Gamma \vdash \langle t_1'', t_2'' \rangle : G_1'' \times G_2''}$$

Finally, using the definition of type precision we can infer that

$$G_1' \times G_2' \sqsubseteq G_1'' \times G_2''$$

and the result holds.

Case (Gasc). Then $t_1 = t :: G_1$. By (Gasc) we know that:

$$(\text{Gasc})\frac{\Xi;\Delta;\Gamma \vdash t : G \qquad \Xi;\Delta \vdash G \sim G_1}{\Xi;\Delta;\Gamma \vdash t :: G_1 : G_1} \tag{7}$$

Consider t_2 such that $t_1 \sqsubseteq t_2$. By definition of term precision t_2 must have the form $t_2 = t' :: G_2$ and therefore

$$(\text{Pasc})\frac{t \sqsubseteq t' \qquad G_1 \sqsubseteq G_2}{t :: G_1 \sqsubseteq t' :: G_2} \tag{8}$$

Using induction hypotheses on the premises of (7) and (8), $\Xi;\Delta;\Gamma \vdash t' : G'$ where $G \sqsubseteq G'$. We can use rule (Gasc) and Lemma 2.13 to derive:

$$(\text{Gasc})\frac{\Xi;\Delta;\Gamma \vdash t' : G' \qquad \Xi;\Delta \vdash G' \sim G_2}{\Xi;\Delta;\Gamma \vdash t' :: G_2 : G_2}$$

Where $G_1 \sqsubseteq G_2$ and the result holds.

Case (Cop). Then $t_1 = op(\overline{t})$ and $G_1 = G^*$. By (Gop) we know that:

$$(\text{Gop})\frac{\begin{array}{c}\Xi;\Delta;\Gamma \vdash \overline{t} : \overline{G} \qquad ty(op) = \overline{G_2} \rightarrow G^* \\ \Xi;\Delta \vdash \overline{G} \sim \overline{G_2}\end{array}}{\Xi;\Delta;\Gamma \vdash op(\overline{t}) : G^*} \tag{9}$$

Consider t_2 such that $t_1 \sqsubseteq t_2$. By definition of term precision t_2 must have the form $t_2 = op(\overline{t'})$ and therefore

$$\text{(Pop)}\,\frac{\overline{t} \sqsubseteq \overline{t'}}{op(\overline{t}) \sqsubseteq op(\overline{t'})} \tag{10}$$

Using induction hypotheses on the premises of (9) and (10), $\Xi; \Delta; \Gamma \vdash \overline{t'} : \overline{G'}$, where $\overline{G} \sqsubseteq \overline{G'}$. Using the Lemma 2.13 we know that $\Xi; \Delta \vdash \overline{G'} \sim \overline{G_2}$. Therefore we can use rule (Gop) to derive:

$$\text{(Gop)}\,\frac{\begin{array}{c}\Xi; \Delta; \Gamma \vdash \overline{t'} : \overline{G'} \qquad ty(op) = \overline{G_2} \rightarrow G^* \\ \Xi; \Delta \vdash \overline{G'} \sim \overline{G_2}\end{array}}{\Xi; \Delta; \Gamma \vdash op(\overline{t'}) : G^*}$$

and the result holds.

Case (Gapp). Then $t_1 = t'_1\ t'_2$ and $G_1 = cod^\sharp(G'_1)$. By (Gapp) we know that:

$$\text{(Gapp)}\,\frac{\begin{array}{c}\Xi; \Delta; \Gamma \vdash t'_1 : G'_1 \qquad \Xi; \Delta; \Gamma \vdash t'_2 : G'_2 \\ \Xi; \Delta \vdash dom^\sharp(G'_1) \sim G'_2\end{array}}{\Xi; \Delta; \Gamma \vdash t'_1\ t'_2 : cod^\sharp(G'_1)} \tag{11}$$

Consider t_2 such that $t_1 \sqsubseteq t_2$. By definition of term precision t_2 must have the form $t_2 = t''_1\ t''_2$ and therefore

$$\text{(Papp)}\,\frac{t'_1 \sqsubseteq t''_1 \qquad t'_2 \sqsubseteq t''_2}{t'_1\ t'_2 \sqsubseteq t''_1\ t''_2} \tag{12}$$

Using induction hypotheses on the premises of (11) and (12), $\Xi; \Delta; \Gamma \vdash t''_1 : G''_1$ and $\Xi; \Delta; \Gamma \vdash t''_2 : G''_2$, where $G'_1 \sqsubseteq G''_1$ and $G'_2 \sqsubseteq G''_2$. By definition type precision and the definition of $dom^\sharp$, $dom^\sharp(G'_1) \sqsubseteq dom^\sharp(G''_1)$ and, therefore by Lemma 2.13, $\Xi; \Delta \vdash dom^\sharp(G''_1) \sim G''_2$. Also, by the previous argument $cod^\sharp(G'_1) \sqsubseteq cod^\sharp(G''_1)$. Then we can use rule (Gapp) to derive:

$$\text{(Gapp)}\,\frac{\begin{array}{c}\Xi; \Delta; \Gamma \vdash t''_1 : G''_1 \qquad \Xi; \Delta; \Gamma \vdash t''_2 : G''_2 \\ \Xi; \Delta \vdash dom^\sharp(G''_1) \sim G''_2\end{array}}{\Xi; \Delta; \Gamma \vdash t''_1\ t''_2 : cod^\sharp(G''_1)}$$

and the result holds.

Case (GappG). Then $t_1 = t\ [G]$. By (GappG) we know that:

$$\text{(GappG)}\,\frac{\Xi; \Delta; \Gamma \vdash t : G'_1 \qquad \Xi; \Delta \vdash G}{\Xi; \Delta; \Gamma \vdash t\ [G] : inst^\sharp(G'_1, G)} \tag{13}$$

where $G_1 = inst^\sharp(G'_1, G)$. Consider t_2 such that $t_1 \sqsubseteq t_2$. By definition of term precision t_2 must have the form $t_2 = t'\ [G']$ and therefore

$$\text{(PappG)}\,\frac{t \sqsubseteq t' \qquad G \sqsubseteq G'}{t\ [G] \sqsubseteq t'\ [G']} \tag{14}$$

Using induction hypotheses on the premises of (13) and (14), $\Xi; \Delta; \Gamma \vdash t' : G'_2$ where $G'_1 \sqsubseteq G'_2$. We can use rule (GappG) and Lemma 2.2 to derive:

$$\text{(Gasc)}\,\frac{\Xi; \Delta; \Gamma \vdash t' : G'_2 \qquad\qquad \Xi; \Delta \vdash G'}{\Xi; \Delta; \Gamma \vdash t'\ [G'] : inst^\sharp(G'_2, G')}$$

Finally, by the Lemma 2.15 we know that $inst^\sharp(G'_1, G) \sqsubseteq inst^\sharp(G'_2, G')$ and the result holds.

Case (Gpairi). Then $t_1 = \pi_i(t)$ and $G_1 = proj_i^\sharp(G)$. By (Gpair) we know that:

$$\text{(Gpairi)}\frac{\Xi; \Delta; \Gamma \vdash t : G}{\Xi; \Delta; \Gamma \vdash \pi_i(t) : proj_i^\sharp(G)} \tag{15}$$

Consider t_2 such that $t_1 \sqsubseteq t_2$. By definition of term precision, t_2 must have the form $\pi_i(t')$ and therefore

$$\text{(Ppairi)}\frac{t \sqsubseteq t'}{\pi_i(t) \sqsubseteq \pi_i(t')} \tag{16}$$

Using induction hypotheses on the premises of (15) and (16), $\Xi; \Delta; \Gamma \vdash t' : G'$ where $G \sqsubseteq G'$. Then we can use rule (Gpairi) to derive:

$$\text{(Gpairi)}\frac{\Xi; \Delta; \Gamma \vdash t' : G'}{\Xi; \Delta; \Gamma \vdash \pi_i(t') : proj_i^\sharp(G')}$$

Finally, by the Lemma 2.16 we can infer that $proj_i^\sharp(G) \sqsubseteq proj_i^\sharp(G')$ and the result holds.

□

Proposition 6.10 (Static gradual guarantee). *Let t and t' be closed GSF terms such that $t \sqsubseteq t'$ and $\vdash t : G$. Then $\vdash t' : G'$ and $G \sqsubseteq G'$.*

Proof. Direct corollary of Prop. 2.17. □

3 GSF: DYNAMICS

In this section, we expose auxiliary definitions of the dynamic semantics of GSF. First, we present type precision, interior and consistent transitivity definitions for evidence types. Then we show some important definitions, used in the dynamic semantics of GSFε. Finally, we present the translation semantics from GSF to GSFε.

3.1 Evidence Type Precision

Figure 20 presents the definition of the evidence type precision.

$\boxed{E \sqsubseteq E}$ **Type precision**

$$\frac{}{B \sqsubseteq B} \qquad \frac{}{X \sqsubseteq X} \qquad \frac{E_1 \sqsubseteq E_1' \quad E_2 \sqsubseteq E_2'}{E_1 \rightarrow E_2 \sqsubseteq E_1' \rightarrow E_2'} \qquad \frac{E_1 \sqsubseteq E_2}{\forall X.E_1 \sqsubseteq \forall X.E_2}$$

$$\frac{E_1 \sqsubseteq E_1' \quad E_2 \sqsubseteq E_2'}{E_1 \times E_2 \sqsubseteq E_1' \times E_2'} \qquad \frac{E_1 \sqsubseteq E_2}{\alpha^{E_1} \sqsubseteq \alpha^{E_2}} \qquad \frac{}{E \sqsubseteq ?}$$

Fig. 20. Evidence Type Precision

3.2 Initial Evidence

In Figure 21 we present the interior function, used to compute the initial evidence.

$$\boxed{\mathcal{I} : \text{ETYPE} \times \text{ETYPE} \rightharpoonup \text{EVIDENCE}}$$

$$\frac{E \in \text{BASETYPE} \cup \text{TYPEVAR} \cup \{?\}}{\mathcal{I}(E, E) = \mathcal{I}(?, E) = \mathcal{I}(E, ?) = \langle E, E\rangle} \qquad \frac{\mathcal{I}(E_1, E_2) = \langle E_1', E_2'\rangle}{\mathcal{I}(\alpha^{E_1}, E_2) = \langle \alpha^{E_1'}, E_2'\rangle} \qquad \frac{\mathcal{I}(E_1, E_2) = \langle E_1', E_2'\rangle}{\mathcal{I}(E_1, \alpha^{E_2}) = \langle E_1', \alpha^{E_2'}\rangle}$$

$$\frac{\mathcal{I}(E_{11} \rightarrow E_{12}, ? \rightarrow ?) = \langle E_1', E_2'\rangle}{\mathcal{I}(E_{11} \rightarrow E_{12}, ?) = \langle E_1', E_2'\rangle} \qquad \frac{\mathcal{I}(? \rightarrow ?, E_{11} \rightarrow E_{12}) = \langle E_1', E_2'\rangle}{\mathcal{I}(?, E_{11} \rightarrow E_{12}) = \langle E_1', E_2'\rangle}$$

$$\frac{\mathcal{I}(\forall X.E, \forall X.?) = \langle E_1', E_2'\rangle}{\mathcal{I}(\forall X.E, ?) = \langle E_1', E_2'\rangle} \qquad \frac{\mathcal{I}(\forall X.?, \forall X.E) = \langle E_1', E_2'\rangle}{\mathcal{I}(?, \forall X.E) = \langle E_1', E_2'\rangle}$$

$$\frac{\mathcal{I}(E_{11} \times E_{12}, ? \times ?) = \langle E_{11}' \times E_{12}', E_{21}' \times E_{22}'\rangle}{\mathcal{I}(E_{11} \times E_{12}, ?) = \langle E_{11}' \times E_{12}', E_{21}' \times E_{22}'\rangle} \qquad \frac{\mathcal{I}(? \times ?, E_{11} \times E_{12}) = \langle E_{11}' \times E_{12}', E_{21}' \times E_{22}'\rangle}{\mathcal{I}(?, E_{11} \times E_{12}) = \langle E_{11}' \times E_{12}', E_{21}' \times E_{22}'\rangle}$$

$$\frac{\mathcal{I}(E_{21}, E_{11}) = \langle E_{21}', E_{11}'\rangle \quad \mathcal{I}(E_{12}, E_{22}) = \langle E_{12}', E_{22}'\rangle}{\mathcal{I}(E_{11} \rightarrow E_{12}, E_{21} \rightarrow E_{22}) = \langle E_{11}' \rightarrow E_{12}', E_{21}' \rightarrow E_{22}'\rangle}$$

$$\frac{\mathcal{I}(E_{11}, E_{21}) = \langle E_{11}', E_{21}'\rangle \quad \mathcal{I}(E_{12}, E_{22}) = \langle E_{12}', E_{22}'\rangle}{\mathcal{I}(E_{11} \times E_{12}, E_{21} \times E_{22}) = \langle E_{11}' \times E_{12}', E_{21}' \times E_{22}'\rangle} \qquad \frac{\mathcal{I}(E_1, E_2) = \langle E_1', E_2'\rangle}{\mathcal{I}(\forall X.E_1, \forall X.E_2) = \langle \forall X.E_1', \forall X.E_2'\rangle}$$

Fig. 21. GSF: Computing Initial Evidence

3.3 Consistent Transitivity

In Figure 22, we present the definition of consistent transitivity for evidence types.

$$\text{(base)}\,\frac{}{\langle B, B\rangle \fatsemi \langle B, B\rangle = \langle B, B\rangle} \qquad \text{(typeVar)}\,\frac{}{\langle X, X\rangle \fatsemi \langle X, X\rangle = \langle X, X\rangle}$$

$$\text{(idL)}\,\frac{}{\langle E_1, E_2\rangle \fatsemi \langle ?, ?\rangle = \langle E_1, E_2\rangle} \qquad \text{(idR)}\,\frac{}{\langle ?, ?\rangle \fatsemi \langle E_1, E_2\rangle = \langle E_1, E_2\rangle}$$

$$\text{(sealL)}\,\frac{\langle E_1, E_2\rangle \fatsemi \langle E_3, E_4\rangle = \langle E_1', E_2'\rangle}{\langle E_1, E_2\rangle \fatsemi \langle E_3, \alpha^{E_4}\rangle = \langle E_1', \alpha^{E_2'}\rangle} \qquad \text{(sealR)}\,\frac{\langle E_1, E_2\rangle \fatsemi \langle E_3, E_4\rangle = \langle E_1', E_2'\rangle}{\langle \alpha^{E_1}, E_2\rangle \fatsemi \langle E_3, E_4\rangle = \langle \alpha^{E_1'}, E_2'\rangle}$$

$$\text{(unsl)}\,\frac{\langle E_1, E_2\rangle \fatsemi \langle E_3, E_4\rangle = \langle E_1', E_2'\rangle}{\langle E_1, \alpha^{E_2}\rangle \fatsemi \langle \alpha^{E_3}, E_4\rangle = \langle E_1', E_2'\rangle}$$

$$\text{(func)}\,\frac{\langle E_{41}, E_{31}\rangle \fatsemi \langle E_{21}, E_{11}\rangle = \langle E_3, E_1\rangle \qquad \langle E_{12}, E_{22}\rangle \fatsemi \langle E_{32}, E_{42}\rangle = \langle E_2, E_4\rangle}{\langle E_{11} \rightarrow E_{12}, E_{21} \rightarrow E_{22}\rangle \fatsemi \langle E_{31} \rightarrow E_{32}, E_{41} \rightarrow E_{42}\rangle = \langle E_1 \rightarrow E_2, E_3 \rightarrow E_4\rangle}$$

$$\text{(abst)}\,\frac{\langle E_1, E_2\rangle \fatsemi \langle E_3, E_4\rangle = \langle E_1', E_2'\rangle}{\langle \forall X.E_1, \forall X.E_2\rangle \fatsemi \langle \forall X.E_3, \forall X.E_4\rangle = \langle \forall X.E_1', \forall X.E_2'\rangle}$$

$$\text{(pair)}\,\frac{\langle E_{11}, E_{21}\rangle \fatsemi \langle E_{31}, E_{41}\rangle = \langle E_1, E_3\rangle \qquad \langle E_{12}, E_{22}\rangle \fatsemi \langle E_{32}, E_{42}\rangle = \langle E_2, E_4\rangle}{\langle E_{11} \times E_{12}, E_{21} \times E_{22}\rangle \fatsemi \langle E_{31} \times E_{32}, E_{41} \times E_{42}\rangle = \langle E_1 \times E_2, E_3 \times E_4\rangle}$$

Fig. 22. GSF: Consistent Transitivity

3.4 GSFε: Dynamic Semantics

In this section, we show the function definitions used in the dynamic semantics of GSFε, specifically in the type application rule (RappG).

Definition 3.1.

$$\varepsilon_{out} \triangleq \langle E_*[\alpha^E], E_*[E']\rangle \qquad \text{where } E_* = lift_{\Xi}(unlift(\pi_2(\varepsilon))), \alpha^E = lift_{\Xi'}(\alpha), E' = lift_{\Xi}(G')$$

Definition 3.2. $\langle E_1, E_2\rangle\,[E_3] = \langle E_1[E_3], E_2[E_3]\rangle$

Definition 3.3.

$$s[\alpha^E/X] = \begin{cases} b & s = b \\ \lambda x : G_1[\alpha/X].t[\alpha^E/X] & s = \lambda x : G_1.t \\ \Lambda Y.t[\alpha^E/X] & s = \Lambda Y.t \\ \langle s_1[\alpha^E/X], s_2[\alpha^E/X]\rangle & s = \langle s_1, s_2\rangle \\ x & s = x \\ \varepsilon[\alpha^E/X]t[\alpha^E/X] :: G[\alpha/X] & s = \varepsilon t :: G \\ op(\overline{t[\alpha^E/X]}) & s = op(\overline{t}) \\ t_1[\alpha^E/X]\ t_2[\alpha^E/X] & s = t_1\ t_2 \\ \pi_i(t[\alpha^E/X]) & s = \pi_i(t) \\ t[\alpha^E/X]\ [G[\alpha/X]] & s = t\ [G] \end{cases}$$

Definition 3.4.

$$lift_\Xi(G) = \begin{cases} lift_\Xi(G_1) \rightarrow lift_\Xi(G_2) & G = G_1 \rightarrow G_2 \\ \forall X.lift_\Xi(G_1) & G = \forall X.G_1 \\ lift_\Xi(G_1) \times lift_\Xi(G_2) & G = G_1 \times G_2 \\ \alpha^{lift_\Xi(\Xi(\alpha))} & G = \alpha \\ G & \text{otherwise} \end{cases}$$

Definition 3.5.

$$unlift(E) = \begin{cases} B & E = B \\ unlift(E_1) \rightarrow unlift(E_2) & E = E_1 \rightarrow E_2 \\ \forall X.unlift(E_1) & E = \forall X.E_1 \\ unlift(E_1) \times unlift(E_2) & E = E_1 \times E_2 \\ \alpha & E = \alpha^{E_1} \\ X & E = X \\ ? & E = ? \end{cases}$$

3.5 Translation from GSF to GSFε

In this section we present the translation from GSF to GSFε (Figure 23), which inserts ascriptions to ensure that top-level constructors match in every elimination form. We use the following normalization metafunction:

$$norm(t, G_1, G_2) = \varepsilon t :: G_2, \text{ where } \varepsilon = \mathcal{I}_\Xi(G_1, G_2)$$
$$\mathcal{I}_\Xi(G_1, G_2) = \mathcal{I}(lift_\Xi(G_1), lift_\Xi(G_2))$$

Lemma 7.1 (Translation Preserves Typing). *Let t be a GSF term. If $\Delta; \Gamma \vdash t : G$ then $\Delta; \Gamma \vdash t \rightsquigarrow t_\varepsilon : G$ and $\Delta; \Gamma \vdash t_\varepsilon : G$.*

Proof. The proof follows by induction on the typing derivation of $\Delta; \Gamma \vdash t : G$. □

$\boxed{\Delta;\Gamma \vdash v \rightsquigarrow_v u : G}$ **Value translation**

$$\text{(G}b\text{)}\frac{ty(b) = B \quad \Delta \vdash \Gamma}{\Delta;\Gamma \vdash b \rightsquigarrow_v b : B} \qquad \text{(Gpairu)}\frac{\Delta;\Gamma \vdash v_1 \rightsquigarrow u_1 : G_1 \quad \Delta;\Gamma \vdash v_2 \rightsquigarrow u_2 : G_2}{\Delta;\Gamma \vdash \langle v_1, v_2\rangle \rightsquigarrow_v \langle u_1, u_2\rangle : G_1 \times G_2}$$

$$\text{(G}\lambda\text{)}\frac{\Delta;\Gamma, x : G \vdash t \rightsquigarrow t' : G'}{\Delta;\Gamma \vdash (\lambda x : G.t) \rightsquigarrow_v (\lambda x : G.t') : G \to G'} \qquad \text{(G}\Lambda\text{)}\frac{\Delta, X;\Gamma \vdash t \rightsquigarrow t' : G \quad \Delta \vdash \Gamma}{\Delta;\Gamma \vdash (\Lambda X.t) \rightsquigarrow_v (\Lambda X.t') : \forall X.G}$$

$\boxed{\Delta;\Gamma \vdash t \rightsquigarrow t : G}$ **Term translation**

$$\text{(Gu)}\frac{\Delta;\Gamma \vdash v \rightsquigarrow_v u : G \quad \varepsilon = \mathcal{I}(G, G)}{\Delta;\Gamma \vdash v \rightsquigarrow \varepsilon u :: G : G} \qquad \text{(Gascu)}\frac{\Delta;\Gamma \vdash v \rightsquigarrow_v u : G \quad \varepsilon = \mathcal{I}(G, G')}{\Delta;\Gamma \vdash v :: G' \rightsquigarrow \varepsilon u :: G' : G'}$$

$$\text{(Gx)}\frac{x : G \in \Gamma \quad \Delta \vdash \Gamma}{\Delta;\Gamma \vdash x \rightsquigarrow x : G} \qquad \text{(Gasct)}\frac{t \neq v \quad \Delta;\Gamma \vdash t \rightsquigarrow t' : G \quad \varepsilon = \mathcal{I}(G, G')}{\Delta;\Gamma \vdash t :: G' \rightsquigarrow \varepsilon t' :: G' : G'}$$

$$\text{(Gpairt)}\frac{(t_1 \neq v_1 \vee t_2 \neq v_2) \quad \Delta;\Gamma \vdash t_1 \rightsquigarrow t_1' : G_1 \quad \Delta;\Gamma \vdash t_2 \rightsquigarrow t_2' : G_2}{\Delta;\Gamma \vdash \langle t_1, t_2\rangle \rightsquigarrow \langle t_1', t_2'\rangle : G_1 \times G_2}$$

$$\text{(Gop)}\frac{\Delta;\Gamma \vdash \overline{t} \rightsquigarrow \overline{t'} : \overline{G_1} \quad ty(op) = \overline{G_2} \to G \quad \overline{t''} = \overline{norm(t', G_1, G_2)}}{\Delta;\Gamma \vdash op(\overline{t}) \rightsquigarrow op(\overline{t''}) : G}$$

$$\text{(Gapp)}\frac{\begin{array}{cc}\Delta;\Gamma \vdash t_1 \rightsquigarrow t_1' : G_1 & t_1'' = norm(t_1', G_1, dom^\sharp(G_1) \to cod^\sharp(G_1)) \\ \Delta;\Gamma \vdash t_2 \rightsquigarrow t_2' : G_2 & t_2'' = norm(t_2', G_2, dom^\sharp(G_1))\end{array}}{\Delta;\Gamma \vdash t_1\ t_2 \rightsquigarrow t_1''\ t_2'' : cod^\sharp(G_2)}$$

$$\text{(GappG)}\frac{\Delta;\Gamma \vdash t \rightsquigarrow t' : G \quad \Delta \vdash G' \quad t'' = norm(t', G, \forall var^\sharp(G).schm_u^\sharp(G))}{\Delta;\Gamma \vdash t\ [G'] \rightsquigarrow t''\ [G'] : inst^\sharp(G, G')}$$

$$\text{(Gpair}i\text{)}\frac{\Delta;\Gamma \vdash t \rightsquigarrow t' : G \quad t'' = norm(t', G, proj_1^\sharp(G) \times proj_2^\sharp(G))}{\Delta;\Gamma \vdash \pi_i(t) \rightsquigarrow \pi_i(t'') : proj_i^\sharp(G)}$$

$$\begin{array}{ll}
var^\sharp : \text{GType} \rightharpoonup \text{GType} & schm_u^\sharp : \text{GType} \rightharpoonup \text{GType} \\
var^\sharp(\forall X.G) = X & schm_u^\sharp(\forall X.G) = G \\
var^\sharp(?) = X \text{ fresh} & schm_u^\sharp(?) = ? \\
var^\sharp(G) \text{ undefined o/w} & schm_u^\sharp(G) \text{ undefined o/w}
\end{array}$$

$$norm(t, G_1, G_2) = \varepsilon t :: G_2, \text{ where } \varepsilon = \mathcal{I}(G_1, G_2)$$

Fig. 23. GSF to GSFε translation.

4 GSF: PROPERTIES

In this section we present some properties of GSF. Section 4.1, presents Type Safety and its proof. Section 4.2, shows the property and proof about static terms do not fail.

4.1 Type Safety

In this section we present the proof of type safety for GSFε.

We define what it means for a store to be well typed with respect to a term. Informally, all free locations of a term and of the contents of the store must be defined in the domain of that store. Also, the store must preserve types between intrinsic locations and underlying values.

Lemma 4.1 (Canonical forms). *Consider a value* $\Xi;\cdot;\cdot \vdash v : G$. *Then* $v = \varepsilon u :: G$, *with* $\Xi;\cdot;\cdot \vdash u : G'$ *and* $\varepsilon \Vdash \Xi \vdash G' \sim G$. *Furthermore:*

1. *If* $G = B$, *then* $v = \varepsilon_B b :: B$, *with* $\Xi;\cdot;\cdot \vdash b : B$ *and* $\varepsilon_B \Vdash \Xi \vdash B \sim B$.
2. *If* $G = G_1 \to G_2$, *then* $v = \varepsilon(\lambda x : G_1'.t) :: G_1 \to G_2$, *with* $\Xi;\cdot;x : G_1' \vdash t : G_2'$ *and* $\varepsilon \Vdash \Xi \vdash G_1' \to G_2' \sim G_1 \to G_2$.
3. *If* $G = \forall X.G_1$, *then* $v = \varepsilon(\Lambda X.t) :: \forall X.G_1$, *with* $\Xi;\Delta,X;\cdot \vdash t : G_1'$ *and* $\varepsilon \Vdash \Xi \vdash \forall X.G_1' \sim \forall X.G_1$.
4. *If* $G = G_1 \times G_2$, *then* $v = \varepsilon\langle u_1, u_2\rangle :: G_1 \times G_2$, *with* $\Xi;\cdot;\cdot \vdash u_1 : G_1'$, $\Xi;\cdot;\cdot \vdash u_2 : G_2'$ *and* $\varepsilon \Vdash \Xi \vdash G_1' \times G_2' \sim G_1 \times G_2$.

Proof. By direct inspection of the formation rules of evidence augmented terms. □

Lemma 4.2 (Substitution). *If* $\Xi;\Delta;\Gamma, x : G_1 \vdash t : G$, *and* $\Xi;\cdot;\cdot \vdash v : G_1$, *then* $\Xi;\Delta;\Gamma \vdash t[v/x] : G$.

Proof. By induction on the derivation of $\Xi;\Delta;\Gamma, x : G_1 \vdash t : G$. □

Lemma 4.3. *If* $\varepsilon \Vdash \Xi;\Delta,X \vdash G_1 \sim G_2$, $\Xi;\cdot \vdash G'$, $\alpha \notin dom(\Xi)$, *and* $E = lift_{\Xi}(G')$, *then* $\varepsilon[\alpha^{E'}/X] \Vdash \Xi, \alpha := G';\Delta \vdash G_1[\alpha/X] \sim G_2[\alpha/X]$.

Proof. By induction on the judgment $\varepsilon \Vdash \Xi;\Delta,X \vdash G_1 \sim G_2$ and the definition of evidences. □

Lemma 4.4 (Type Substitution). *If* $\Xi;\Delta,X;\Gamma \vdash t : G$, $\Xi;\cdot \vdash G'$, $\alpha \notin dom(\Xi)$, *and* $E = lift_{\Xi}(G')$, *then* $\Xi, \alpha := G';\Delta;\Gamma \vdash t[\alpha^E/X] : G[\alpha/X]$.

Proof. By induction on the derivation of $\Xi;\Delta,X;\Gamma \vdash t : G$ and Lemma 4.3. □

Lemma 4.5. *If* $\varepsilon_1 \Vdash \Xi;\Delta \vdash G_1' \sim G_1$, *and* $\varepsilon_2 \Vdash \Xi;\Delta \vdash G_2' \sim G_2$, *then* $\varepsilon_1 \times \varepsilon_2 \Vdash \Xi;\Delta \vdash G_1' \times G_2' \sim G_1 \times G_2$.

Proof. By definition of the judgment $\varepsilon \Vdash \Xi;\Delta,X \vdash G_1' \times G_2' \sim G_1 \times G_2$ and the definition of evidences. □

Lemma 4.6. *If* $\varepsilon \Vdash \Xi;\Delta \vdash G' \sim G$ *then* $p_i(\varepsilon) \Vdash \Xi;\Delta \vdash proj_i^{\sharp}(G') \sim proj_i^{\sharp}(G)$.

Proof. By definition of judgment $\varepsilon \Vdash \Xi;\Delta,X \vdash proj_i^{\sharp}(G') \sim proj_i^{\sharp}(G)$ and the definition of evidences. □

Proposition 4.7 ($\longrightarrow$ is well defined). *If* $\Xi;\cdot;\cdot \vdash t : G$, *then either*

- $\Xi \triangleright t \longrightarrow \Xi' \triangleright t'$, $\Xi \subseteq \Xi'$ *and* $\Xi';\cdot;\cdot \vdash t' : G$; *or*
- $\Xi \triangleright t \longrightarrow$ **error**

Proof. By induction on the structure of a derivation of $\Xi \triangleright t \longrightarrow r$, considering the last rule used in the derivation.

Case (Rapp). Then $t = (\varepsilon_1(\lambda x : G_{11}.t_1) :: G_1 \to G_2)\ (\varepsilon_2 u :: G_1)$. Then

$$(Eapp)\dfrac{(Easc)\dfrac{\dfrac{\Xi;\cdot;x : G_{11} \vdash t_1 : G_{12}}{\Xi;\cdot;\cdot \vdash (\lambda x : G_{11}.t_1) : G_{11} \to G_{12}} \quad \varepsilon_1 \Vdash \Xi;\cdot \vdash G_{11} \to G_{12} \sim G_1 \to G_2}{\Xi;\cdot;\cdot \vdash (\varepsilon_1(\lambda x : G_{11}.t_1) :: G_1 \to G_2) : G} \quad (Easc)\dfrac{\Xi;\cdot;\cdot \vdash u : G_2' \quad \varepsilon_2 \Vdash \Xi;\cdot \vdash G_2' \sim G_1}{\Xi;\cdot;\cdot \vdash (\varepsilon_2 u :: G_1) : G_1}}{\Xi;\cdot;\cdot \vdash (\varepsilon_1(\lambda x : G_{11}.t_1) :: G)\ (\varepsilon_2 u :: G_1) : G_2}$$

If $\varepsilon' = (\varepsilon_2 \mathbin{⨟} dom(\varepsilon_1))$ is not defined, then $\Xi \triangleright t \longrightarrow$ **error**, and then the result hold immediately. Suppose that consistent transitivity does hold, then

$$\Xi \triangleright (\varepsilon_1(\lambda x : G_{11}.t_1) :: G_1 \to G_2)\ (\varepsilon_2 u :: G_1) \longrightarrow \Xi \triangleright cod(\varepsilon_1)(t_1[\varepsilon' u :: G_{11})/x]) :: G_2$$

As $\varepsilon_2 \vdash G_2' \sim G_1$ and by inversion lemma $dom(\varepsilon_1) \vdash G_1 \sim G_{11}$, then $\varepsilon' \vdash G_2' \sim G_{11}$. Therefore $\Xi;\cdot;\cdot \vdash \varepsilon' u :: G_{11} : G_{11}$, and by Lemma 4.2, $\Xi;\cdot;\cdot \vdash t[(\varepsilon' u :: G_{11})/x] : G_{12}$.

Let us call $t'' = t[(\varepsilon' u :: G_{11})/x]$. Then

$$(Easc)\dfrac{\Xi;\cdot;\cdot \vdash t_1[\varepsilon' u :: G_{11})/x] : G_{12} \qquad cod(\varepsilon_1) \Vdash \Xi;\cdot \vdash G_{12} \sim G_2}{\Xi;\cdot;\cdot \vdash cod(\varepsilon_1)(t_1[\varepsilon' u :: G_{11})/x]) :: G_2 : G_2}$$

and the result holds.

Case (RappG). Then $t = (\varepsilon \Lambda X.t_1 :: \forall X.G_x)\ [G']$. Consider $G_x = schm^{\sharp}_u(G)$, then

$$(EappG)\dfrac{(Easc)\dfrac{\Xi;X;\cdot \vdash t_1 : G_1 \qquad \varepsilon \Vdash \Xi;X;\cdot \vdash G_1 \sim \forall X.G_x}{\Xi;\cdot;\cdot \vdash (\varepsilon \Lambda X.t_1 :: \forall X.G_x) : \forall X.G_x} \qquad \Xi;\cdot \vdash G'}{\Xi;\cdot;\cdot \vdash (\varepsilon \Lambda X.t_1 :: \forall X.G_x)\ [G'] : G_x[G'/X]}$$

Then

$$\Xi \triangleright (\varepsilon \Lambda X.t_1 :: G)\ [G'] \longrightarrow \Xi' \triangleright \varepsilon_G^{E'/\alpha^{E'}}(\varepsilon[\alpha^{E'}]t_1[\alpha^{E'}/X] :: G_x[\alpha/X]) :: G_x[G'/X]$$

where $\Xi' \triangleq \Xi, \alpha := G', \alpha \notin dom(\Xi)$, and $E' \triangleq lift_{\Xi}(G')$, and $\varepsilon_{\forall X.G_x}^{E'/\alpha^{E'}} = \langle lift_{\Xi}(G_x)[\alpha^{E'}/X], lift_{\Xi}(G_x[G'/X])\rangle$. Notice that $\langle lift_{\Xi}(G_x[\alpha/X]), lift_{\Xi}(G_x[G'/X])\rangle = \mathcal{I}(G_x[\alpha/X], G_x[G'/X])$, and by definition of the special substitution, $lift_{\Xi}(G_x)[\alpha^{E'}/X] \sqsubseteq lift_{\Xi}(G_x[\alpha/X])$ (because $lift_{\Xi}(\alpha) = \alpha^{E'}$, and the substitution on evidences just extend unknowns with α). Therefore $\varepsilon_{\forall X.G_x}^{E'/\alpha^{E'}} \sqsubseteq \mathcal{I}(G_x[\alpha/X], G_x[G'/X])$, and $\varepsilon_{\forall X.G_x}^{E'/\alpha^{E'}} \Vdash \Xi;\cdot \vdash G_x[\alpha/X] \sim G_x[G'/X]$. Also by Lemma 4.3 $\varepsilon[\alpha^{E'}] \Vdash \Xi;\cdot \vdash G_1[\alpha/X] \sim G_x[\alpha/X]$, and by Lemma 4.4, $\Xi;\cdot;\cdot \vdash t_1[\alpha^{E'}/X] : G_1[\alpha/X]$.

Then, as $\Xi \subseteq \Xi'$,

$$(Easc)\dfrac{(Easc)\dfrac{\Xi;\cdot;\cdot \vdash t_1[\alpha^{E'}/X] : G_1[\alpha/X] \quad \varepsilon[\alpha^{E'}] \Vdash \Xi;\cdot \vdash G_1[\alpha/X] \sim G_x[\alpha/X]}{\Xi;\cdot;\cdot \vdash (\varepsilon[\alpha^{E'}]t_1[\alpha^{E'}/X] :: G_x[\alpha/X]) : G_x[\alpha/X]} \qquad \varepsilon_G^{E'/\alpha^{E'}} \Vdash \Xi;\cdot \vdash G_x[\alpha/X] \sim G_x[G'/X]}{\Xi;\cdot;\cdot \vdash \varepsilon_G^{E'/\alpha^{E'}}(\varepsilon[\alpha^{E'}]t_1[\alpha^{E'}/X] :: G_x[\alpha/X]) :: G_x[G'/X] : G_x[G'/X]}$$

and the result holds.

Case (*R*asc). Then $t = \varepsilon_1(\varepsilon_2 u :: G_2) :: G$. Then

$$\dfrac{\dfrac{\Xi;\cdot;\cdot \vdash u : G_u \quad \varepsilon_2 \Vdash \Xi;\cdot \vdash G_u \sim G_2}{\Xi;\cdot;\cdot \vdash \varepsilon_2 u :: G_2 : G_2}\,(E\text{asc}) \quad \varepsilon_1 \Vdash \Xi;\cdot \vdash G_2 \sim G}{\Xi;\cdot;\cdot \vdash \varepsilon_1(\varepsilon_2 u :: G_2) :: G : G}\,(E\text{asc})$$

If $(\varepsilon_2 \circ\!\!\!\!\;\, \varepsilon_1)$ is not defined, then $\Xi \triangleright t \longrightarrow$ **error**, and then the result hold immediately. Suppose that consistent transitivity does hold, then

$$\Xi \triangleright \varepsilon_1(\varepsilon_2 u :: G_2) :: G \longrightarrow \Xi \triangleright (\varepsilon_2 \mathbin{⨟} \varepsilon_1)u :: G$$

where $(\varepsilon_2 \mathbin{⨟} \varepsilon_1) \Vdash \Xi;\cdot \vdash G_u \sim G$. Then

$$\dfrac{\Xi;\cdot;\cdot \vdash u : G_u \quad (\varepsilon_2 \mathbin{⨟} \varepsilon_1) \Vdash \Xi;\cdot \vdash G_u \sim G}{\Xi;\cdot;\cdot \vdash (\varepsilon_2 \mathbin{⨟} \varepsilon_1)u :: G : G}\,(E\text{asc})$$

and the result follows.

Case (*R*op). Then $t = op(\overline{\varepsilon u :: B'})$. Then

$$\dfrac{\dfrac{\overline{\Xi;\cdot;\cdot \vdash u : G_u} \quad \overline{\varepsilon \Vdash \Xi;\cdot \vdash G_u \sim B'}}{\Xi;\Delta;\Gamma \vdash \overline{\varepsilon u :: B'} : \overline{B'}}\,(E\text{asc}) \quad ty(op) = \overline{B'} \to B}{\Xi;\cdot;\cdot \vdash op(\overline{\varepsilon u :: B'}) : B}\,(E\text{op})$$

Let us assume that $ty(op) : \overline{B'} \to B$.

$$\Xi \triangleright op(\overline{\varepsilon u :: B'}) \longrightarrow \Xi \triangleright \varepsilon_B\ \delta(op, \overline{u}) :: B$$

But as $\varepsilon_B \vdash \Xi;\cdot \vdash B \sim B$, then

$$\dfrac{\Xi;\cdot;\cdot \vdash \delta(op, \overline{u}) : B \quad \varepsilon_B \Vdash \Xi;\cdot \vdash B \sim B}{\Xi;\cdot;\cdot \vdash \varepsilon_B\ \delta(op, \overline{u}) :: B : B}\,(E\text{asc})$$

and the result follows.

Case (*R*pair). Then $t = \langle \varepsilon_1 u_1 :: G_1, \varepsilon_2 u_2 :: G_2 \rangle$. Then

$$\dfrac{\dfrac{\begin{array}{c}\Xi;\cdot;\cdot \vdash u_1 : G_1' \\ \varepsilon_1 \Vdash \Xi;\cdot \vdash G_1' \sim G_1\end{array}}{\Xi;\cdot;\cdot \vdash \varepsilon_1 u_1 :: G_1}\,(E\text{asc}) \quad \dfrac{\begin{array}{c}\Xi;\cdot;\cdot \vdash u_2 : G_2' \\ \varepsilon_2 \Vdash \Xi;\cdot \vdash G_2' \sim G_2\end{array}}{\Xi;\cdot;\cdot \vdash \varepsilon_2 u_2 :: G_2}\,(E\text{asc})}{\Xi;\cdot;\cdot \vdash \langle \varepsilon_1 u_1 :: G_1, \varepsilon_2 u_2 :: G_2 \rangle : G_1 \times G_2}\,(E\text{pair})$$

Then

$$\Xi \triangleright \langle \varepsilon_1 u_1 :: G_1, \varepsilon_2 u_2 :: G_2 \rangle \longrightarrow \Xi \triangleright (\varepsilon_1 \times \varepsilon_2)\langle u_1, u_2 \rangle :: G_1 \times G_2$$

By Lemma 4.5, $\varepsilon_1 \times \varepsilon_2 \Vdash \Xi;\cdot \vdash G_1' \times G_2' \sim G_1 \times G_2$. Then

$$\dfrac{\dfrac{\Xi;\cdot;\cdot \vdash u_1 : G_1' \quad \Xi;\cdot;\cdot \vdash u_2 : G_2'}{\Xi;\cdot;\cdot \vdash \langle u_1, u_2 \rangle : G_1' \times G_2'}\,(E\text{pair}) \quad \varepsilon_1 \times \varepsilon_2 \Vdash \Xi;\cdot \vdash G_1' \times G_2' \sim G_1 \times G_2}{\Xi;\cdot;\cdot \vdash (\varepsilon_1 \times \varepsilon_2)\langle u_1, u_2 \rangle :: G_1 \times G_2 : G_1 \times G_2}\,(E\text{asc})$$

and the result holds.

Case (*R*proj*i*). Then $t = \pi_i(\varepsilon\langle u_1, u_2 \rangle :: G)$. Then

$$
(E\text{pairi})\dfrac{(E\text{asc})\dfrac{\dfrac{\Xi;\cdot;\cdot\vdash u_i : G'_i}{\Xi;\cdot;\cdot\vdash\langle u_1,u_2\rangle : G'_1\times G'_2}\quad \varepsilon\Vdash\Xi;\cdot\vdash G'_1\times G'_2\sim G}{\varepsilon\langle u_1,u_2\rangle :: G}}{\Xi;\cdot;\cdot\vdash\pi_i(\varepsilon\langle u_1,u_2\rangle :: G) : proj^{\sharp}_i(G)}
$$

Then

$$
\Xi\triangleright\pi_i(\varepsilon\langle u_1,u_2\rangle :: G)\longrightarrow\ \Xi\triangleright p_i(\varepsilon)u_i :: proj^{\sharp}_i(G)
$$

By Lemma 4.6, $p_i(\varepsilon)\Vdash\Xi;\cdot\vdash proj^{\sharp}_i(G'_1\times G'_2)\sim proj^{\sharp}_i(G)$. Then

$$
(E\text{asc})\dfrac{\Xi;\cdot;\cdot\vdash u_i : G'_i\qquad p_i(\varepsilon)\Vdash\Xi;\cdot\vdash proj^{\sharp}_i(G'_1\times G'_2)\sim proj^{\sharp}_i(G)}{\Xi;\cdot;\cdot\vdash p_i(\varepsilon)u_i :: proj^{\sharp}_i(G) : proj^{\sharp}_i(G)}
$$

and the result holds.

□

Proposition 4.8 ($\longmapsto$ is well defined). *If $\Xi;\cdot;\cdot\vdash t : G$, then either*

- $\Xi\triangleright t\longmapsto\Xi'\triangleright t'$, $\Xi\subseteq\Xi'$ *and* $\Xi';\cdot;\cdot\vdash t' : G$*; or*
- $\Xi\triangleright t\longmapsto\textbf{error}$

Proof. By induction on the structure of t.

- If t has some of this form: $\varepsilon_2(\varepsilon_1 u :: G_1) :: G_2$, $op(\overline{\varepsilon u :: G})$, $(\lambda x : G_{11}.t) :: G_1\to G_2)$ $(\varepsilon_2 u :: G_1)$, $\langle\varepsilon_1 u_1 :: G_1, \varepsilon_2 u_2 :: G_2\rangle$, $\pi_i(\varepsilon\langle u_1,u_2\rangle :: G_1\times G_2)$ or $(\varepsilon\Lambda X.t :: \forall X.G)$ $[G']$, then by well-definedness of $\longrightarrow$ (Prop 4.7), $\Xi\triangleright t\ \longrightarrow\ \Xi'\triangleright t'$ and $\Xi\subseteq\Xi'$ and $\Xi';\cdot;\cdot\vdash t' : G$ or $\Xi\triangleright t\ \longrightarrow\ \textbf{error}$, .
 If $\Xi\triangleright t\ \longrightarrow\ \Xi'\triangleright t'$, $\Xi\subseteq\Xi'$ and $\Xi';\cdot;\cdot\vdash t' : G$, then by the rule R$\longrightarrow$ the result holds.
 If $\Xi\triangleright t\ \longrightarrow\ \textbf{error}$, then by the rule Rerr $\Xi\triangleright t\longmapsto\textbf{error}$ and the result holds immediately.
- If $t = f[t_1]$, we know that $\Xi;\cdot;\cdot\vdash f[t_1] : G$ and $\Xi;\cdot;\cdot\vdash t_1 : G'$, where $f : G'\to G$. Then, by the induction hypothesis $\Xi\triangleright t_1\longmapsto\Xi'\triangleright t'_1$, $\Xi\subseteq\Xi'$ and $\Xi';\cdot;\cdot\vdash t'_1 : G$ or $\Xi\triangleright t_1\longmapsto\Xi'\triangleright\textbf{error}$.
 If $\Xi\triangleright t_1\longmapsto\Xi'\triangleright t'_1$, by the R$f$ rule the result holds.
 If $\Xi\triangleright t_1\longmapsto\Xi'\triangleright\textbf{error}$, by the R$f$err rule the result holds. .

□

Proposition 4.9 ($\longmapsto$ is well defined). *If $\Xi;\cdot;\cdot\ \vdash\ t\ :\ G$, $t\ \rightsquigarrow\ t_\varepsilon$, then t_ε is a value v; or $\Xi\triangleright t_\varepsilon\longmapsto\Xi'\triangleright t'_\varepsilon$, $\Xi\subseteq\Xi'$ and $\Xi';\cdot;\cdot\vdash t'_\varepsilon : G$; or $\Xi\triangleright t_\varepsilon\longmapsto$* **error**.

Proof. By induction on the structure of t, using Lemma 4.8 and Canonical Forms (Lemma 4.1). □

Now we can establish type safety of GSF: programs of GSF do not get stuck, though they may terminate with cast errors. Also the store of a program is well typed.

Proposition 8.4 (Type Safety). *If $\vdash t : G$ then either $t\Downarrow\Xi\triangleright v$ with $\Xi\triangleright v : G$, $t\Downarrow$* **error**, *or $t\Uparrow$.*

Proof. Direct by 4.9. □

4.2 Static Terms Do Not Fail

Lemma 8.2. *(Properties of consistent transitivity).*
(a) Associativity. $(\varepsilon_1 \mathbin{⨟} \varepsilon_2) \mathbin{⨟} \varepsilon_3 = \varepsilon_1 \mathbin{⨟} (\varepsilon_2 \mathbin{⨟} \varepsilon_3)$*, or both are undefined.*
(b) Optimality. If $\varepsilon = \varepsilon_1 \mathbin{⨟} \varepsilon_2$ *is defined, then* $\pi_1(\varepsilon) \sqsubseteq \pi_1(\varepsilon_1)$ *and* $\pi_2(\varepsilon) \sqsubseteq \pi_2(\varepsilon_2)$*.*
(c) Monotonicity. If $\varepsilon_1 \sqsubseteq \varepsilon_1{}'$ *and* $\varepsilon_2 \sqsubseteq \varepsilon_2{}'$ *and* $\varepsilon_1 \mathbin{⨟} \varepsilon_2$ *is defined, then* $\varepsilon_1 \mathbin{⨟} \varepsilon_2 \sqsubseteq \varepsilon_1{}' \mathbin{⨟} \varepsilon_2{}'$*.*

Proof. A direct result of the application of the AGT framework. □

Lemma 4.10. *If* ε_1 *and* ε_2 *two static evidences, such that* $\varepsilon_1 \Vdash \Xi; \Delta \vdash T_1 \sim T_2$ *and* $\varepsilon_2 \Vdash \Xi; \Delta \vdash T_2 \sim T_3$*, then* $\varepsilon_1 \mathbin{⨟} \varepsilon_2 = \langle p_1(\varepsilon_1), p_2(\varepsilon_2)\rangle$*.*

Proof. Straightforward induction on types T_1, T_2, T_3 ($\Xi; \Delta \vdash T_2 \sim T_3$ coincides with $\Xi; \Delta \vdash T_2 = T_3$), and optimality of evidences (Lemma 8.2), because the resulting evidence cannot gain precision as each component of the evidences are static (note that precision $\cdot \sqsubseteq \cdot$ between static types coincide with equality of static types $\Xi; \Delta \vdash \cdot = \cdot$). □

Lemma 4.11. *Let* T_1 *and* T_2 *two static types, and* Ξ *a static store, such that* $\Xi; \Delta \vdash T_1 \sim T_2$*. Then* $\mathcal{I}(T_1, T_2) = \mathcal{I}(lift_\Xi(T_1), lift_\Xi(T_2)) = \langle lift_\Xi(T_1), lift_\Xi(T_2)\rangle$*.*

Proof. Straightforward induction on types T_1, T_2, and noticing that we cannot gain precision from the types. □

Proposition 4.12 (Static terms progress and Preservation). *Let* t *be a static term,* Ξ *a static store* ($\Xi = \Sigma$)*, and* G *a static type* ($G = T$)*. If* $\Sigma; \cdot; \cdot \vdash t : T$ *, then either* $\Sigma \triangleright t \longmapsto \Sigma' \triangleright t'$ *and* $\Sigma'; \cdot; \cdot \vdash t' : T$ *, for some* Σ' *and* t' *static; or* t *is a value* v*.*

Proof. By induction on the structure of a derivation of $\Sigma; \cdot; \cdot \vdash t : T$.
Note that $\Xi; \Delta \vdash T_1 \sim T_2$ coincides with $\Xi; \Delta \vdash T_1 = T_2$, so we use the latter notation throughout the proof.

Case ($t = \varepsilon u :: G$). The result is trivial as t is a value.

Case ($t = (\varepsilon_1(\lambda x : T_{11}.t_1) :: T_1 \to T_2)\ (\varepsilon_2 u :: T_1)$). Then

$$\dfrac{(E\text{asc})\dfrac{\dfrac{\Xi; \cdot; x : T_{11} \vdash t_1 : T_{12}}{\Xi; \cdot; \cdot \vdash (\lambda x : T_{11}.t_1) : T_{11} \to T_{12}} \quad \varepsilon_1 \Vdash \Sigma; \Delta \vdash T_{11} \to T_{12} = T_1 \to T_2}{\Xi; \cdot; \cdot \vdash (\varepsilon_1(\lambda x : T_{11}.t_1) :: T_1 \to T_2) : T_1 \to T_2} \quad (E\text{asc})\dfrac{\Xi; \cdot; \cdot \vdash u : T_2' \quad \varepsilon_2 \Vdash \Sigma; \Delta \vdash T_2' = T_1}{\Xi; \cdot; \cdot \vdash (\varepsilon_2 u :: T_1) : T_1}}{\Xi; \cdot; \cdot \vdash (\varepsilon_1(\lambda x : T_{11}.t_1) :: T_1 \to T_2)\ (\varepsilon_2 u :: T_1) : T_2}(E\text{app})$$

By Lemma 4.10, $\varepsilon' = (\varepsilon_2 \mathbin{⨟} dom(\varepsilon_1))$ is defined and by Lemma 4.11, the new evidence is also static. Then

$$\Xi \triangleright (\varepsilon_1(\lambda x : T_{11}.t_1) :: T)\ (\varepsilon_2 u :: T_1) \longrightarrow \Xi \triangleright cod(\varepsilon_1)(t_1[\varepsilon' u :: T_{11})/x]) :: T_2$$

And the result holds immediately by the Lemma 4.2 and the typing rule (Easc).

Case ($t = (\varepsilon \Lambda X.t_1 :: \forall X.T_x)\ [T']$). Then

$$\dfrac{(E\text{asc})\dfrac{\Xi; X; \cdot \vdash t_1 : T_1 \quad \varepsilon \Vdash \Sigma; \Delta \vdash [= \Xi; X; \cdot]T_1 \forall X.T_x}{\Xi; \cdot; \cdot \vdash (\varepsilon \Lambda X.t_1 :: \forall X.T_x) : T} \quad \Xi; \cdot \vdash T'}{\Xi; \cdot; \cdot \vdash (\varepsilon \Lambda X.t_1 :: \forall X.T_x)\ [T'] : T_x[T'/X]}(E\text{app}T)$$

Then

$$(\varepsilon\Lambda X.t_1 :: \forall X.T_x)\ [T'] \longrightarrow \Xi' \triangleright \varepsilon^{E'/\alpha^{E'}}_{\forall X.T_x}(\varepsilon[\alpha^{E'}]t_1[\alpha^{E'}/X] :: T_x[\alpha/X]) :: T_x[T'/X]$$

where $\Xi' \triangleq \Xi, \alpha := T', \alpha \notin dom(\Xi)$, and $E' \triangleq lift_\Xi(T')$, and
$\varepsilon^{E'/\alpha^{E'}}_{\forall X.T_x} = \langle lift_\Xi(T_x)[\alpha^{E'}/X], lift_\Xi(T_x[T'/X])\rangle$. Then, $\Xi \subseteq \Xi'$, and Ξ' is extended with a type name that maps to a static type. Finally, the result holds immediately by the Lemma 4.4 and Lemma 4.3, and the typing rule (Easc).

Case ($t = \Xi \triangleright \varepsilon_1(\varepsilon_2 u :: T_2) :: T$). Then

$$\dfrac{\dfrac{\Xi;\cdot;\cdot \vdash u : T_u \qquad \varepsilon_2 \Vdash \Sigma;\Delta \vdash T_u = T_2}{\Xi;\cdot;\cdot \vdash \varepsilon_2 u :: T_2 : T_2}(Easc) \qquad \varepsilon_1 \Vdash \Sigma;\Delta \vdash T_2 = T}{\Xi;\cdot;\cdot \vdash \varepsilon_1(\varepsilon_2 u :: T_2) :: T : T}(Easc)$$

By Lemma 4.10, $\varepsilon_2 \mathbin{;} \varepsilon_1$ is defined and by Lemma 4.11, the new evidence is also static. Then

$$\Xi \triangleright \varepsilon_1(\varepsilon_2 u :: T_2) :: T \longrightarrow \Xi \triangleright (\varepsilon_2 \mathbin{;} \varepsilon_1)u :: T$$

and the result holds by the typing rule (Easc).

Case ($t = op(\overline{\varepsilon u :: B'})$). Then

$$\dfrac{\dfrac{\overline{\Xi;\cdot;\cdot \vdash u : T_u} \qquad \overline{\varepsilon \Vdash \Sigma;\Delta \vdash T_u = B'}}{\Xi;\Delta;\Gamma \vdash \overline{\varepsilon u :: B'} : \overline{B'}}(Easc) \qquad ty(op) = \overline{B'} \to B}{\Xi;\cdot;\cdot \vdash op(\overline{\varepsilon u :: B'}) : B}(Easc)$$

Let us assume that $ty(op) : \overline{B'} \to B$. Then

$$\Xi \triangleright op(\overline{\varepsilon u :: B'}) \longrightarrow \Xi \triangleright \varepsilon_B\ \delta(op, \overline{u}) :: B$$

And the result holds by the typing rule (Easc).

Case ($t = \langle \varepsilon_1 u_1 :: T_1, \varepsilon_2 u_2 :: T_2\rangle$). Then

$$\dfrac{\dfrac{\Xi;\cdot;\cdot \vdash u_1 : T_1' \quad \varepsilon_1 \Vdash \Sigma;\Delta \vdash T_1' = T_1}{\Xi;\cdot;\cdot \vdash \varepsilon_1 u_1 :: T_1}(Easc) \qquad \dfrac{\Xi;\cdot;\cdot \vdash u_2 : T_2' \quad \varepsilon_2 \Vdash \Sigma;\Delta \vdash T_2' = T_2}{\Xi;\cdot;\cdot \vdash \varepsilon_2 u_2 :: T_2}(Easc)}{\Xi;\cdot;\cdot \vdash \langle \varepsilon_1 u_1 :: T_1, \varepsilon_2 u_2 :: T_2\rangle : T_1 \times T_2}(Epair)$$

Then

$$\Xi \triangleright \langle \varepsilon_1 u_1 :: T_1, \varepsilon_2 u_2 :: T_2\rangle \longrightarrow \Xi \triangleright (\varepsilon_1 \times \varepsilon_2)\langle u_1, u_2\rangle :: T_1 \times T_2$$

and the result holds by the Lemma 4.5.

Case ($t = \pi_i(\varepsilon\langle u_1, u_2\rangle :: T)$). Then

$$\dfrac{\dfrac{\dfrac{\Xi;\cdot;\cdot \vdash u_i : T_i'}{\Xi;\cdot;\cdot \vdash \langle u_1, u_2\rangle : T_1' \times T_2'} \qquad \varepsilon \Vdash \Sigma;\Delta \vdash T_1' \times T_2' = T}{\varepsilon\langle u_1, u_2\rangle :: T}(Easc)}{\Xi;\cdot;\cdot \vdash \pi_i(\varepsilon\langle u_1, u_2\rangle :: T) : proj_i^\sharp(T)}(Epair)$$

Then

$$\Xi \triangleright \pi_i(\varepsilon\langle u_1, u_2\rangle :: T) \longrightarrow \Xi \triangleright p_i(\varepsilon)u_i :: proj_i^\sharp(T)$$

And the result holds by Lemma 4.6.

Case ($t = t_1\ t_2$). Then by induction hypothesis $\Xi \triangleright t_1 \longmapsto \Xi \triangleright t_1'$, and t_1' is static, and so $t_1'\ t_2$.

Case ($t = v\ t_2$). Then by induction hypothesis $\Xi \triangleright t_2 \longmapsto \Xi \triangleright t_2'$, and t_2' is static, and so $v\ t_2'$.

Case ($t = t_1[T]$, $t = \langle t_1, t_2 \rangle$, $t = op(\overline{t_1})$, $t = \pi_i(t_1)$). Similar inductive reasoning to application cases.

□

Proposition 8.5 (Static terms do not fail). *Let t be a static term. If $\vdash t : T$ then $\neg(t \Downarrow$ **error**$)$.*

Proof. Direct by Lemma 4.12. □

5 GSF AND THE DYNAMIC GRADUAL GUARANTEE

In this section, we prove the weaker variant of the DGG in GSFε and then in GSF. We also present auxiliary definitions and Propositions.

5.1 Evidence Type Precision

This section show the definition of evidence type precision.

$$\frac{}{B \leqslant B} \qquad \frac{}{X \leqslant X} \qquad \frac{}{\alpha \leqslant \alpha} \qquad \frac{}{B \leqslant ?} \qquad \frac{E_1 \to E_2 \leqslant ? \to ?}{E_1 \to E_2 \leqslant ?} \qquad \frac{}{? \leqslant ?}$$

$$\frac{E_1 \leqslant E_3 \quad E_2 \leqslant E_4}{E_1 \to E_2 \leqslant E_3 \to E_4} \qquad \frac{E_1 \leqslant E_2}{\forall X.E_1 \leqslant \forall X.E_2} \qquad \frac{E_1 \leqslant E_2}{\alpha^{E_1} \leqslant \alpha^{E_2}} \qquad \frac{E_1 \leqslant E_3 \quad E_2 \leqslant E_4}{\langle E_1, E_2\rangle \leqslant \langle E_3, E_4\rangle}$$

5.2 Monotonicity of Evidence Transitivity and Instantiation

This section presents the proofs of the monotonicity of evidence transitivity and instantiation proposition.

Proposition 9.3 ($\leqslant$-Monotonicity of Consistent Transitivity). *If $\varepsilon_1 \leqslant \varepsilon_2$, $\varepsilon_3 \leqslant \varepsilon_4$, and $\varepsilon_1 \fatsemi \varepsilon_3$ is defined, then $\varepsilon_1 \fatsemi \varepsilon_3 \leqslant \varepsilon_2 \fatsemi \varepsilon_4$.*

Proof. By definition of consistent transitivity for = and the definition of precision.

Case ($\varepsilon_i = \langle B, B\rangle$). The results follows immediately, due

$$\langle B, B\rangle = (\langle B, B\rangle \fatsemi \langle B, B\rangle) \leqslant (\langle B, B\rangle \fatsemi \langle B, B\rangle = \langle B, B\rangle)$$

Case ([X]- $\varepsilon_i = \langle X, X\rangle$). The results follows immediately, due

$$\langle X, X\rangle = (\langle X, X\rangle \fatsemi \langle X, X\rangle) \leqslant (\langle X, X\rangle \fatsemi \langle X, X\rangle = \langle X, X\rangle)$$

Case ([α_1]- $\varepsilon_1 = \langle \alpha^{E_1}, E_1'\rangle$, $\varepsilon_2 = \langle \alpha^{E_2}, E_2'\rangle$, $\varepsilon_3 = \langle E_3, E_3'\rangle$, $\varepsilon_4 = \langle E_4, E_4'\rangle$). By the definition of $\leqslant$, we know that $\langle E_1, E_1'\rangle \leqslant \langle E_2, E_2'\rangle$ and $\langle E_3, E_3'\rangle \leqslant \langle E_4, E_4'\rangle$. By the definition of transitivity we know that $\langle \alpha^{E_1}, E_1'\rangle \fatsemi \langle E_3, E_3'\rangle = \langle \alpha^{E_5}, E_5'\rangle$ and $\langle \alpha^{E_2}, E_2'\rangle \fatsemi \langle E_4, E_4'\rangle = \langle \alpha^{E_6}, E_6'\rangle$, where $\langle E_5, E_5'\rangle = \langle E_1, E_1'\rangle \fatsemi \langle E_3, E_3'\rangle$ and $\langle E_6, E_6'\rangle = \langle E_2, E_2'\rangle \fatsemi \langle E_4, E_4'\rangle$. Therefore, we are required to prove that $\langle \alpha^{E_5}, E_5'\rangle \leqslant \langle \alpha^{E_6}, E_6'\rangle$, or what is the same $\langle E_5, E_5'\rangle \leqslant \langle E_6, E_6'\rangle$. But the result follows immediately by the induction hypothesis on $\langle E_1, E_1'\rangle \leqslant \langle E_2, E_2'\rangle$ and $\langle E_3, E_3'\rangle \leqslant \langle E_4, E_4'\rangle$.

Case ([α_2]- $\varepsilon_1 = \langle E_1, \alpha^{E_1'}\rangle$, $\varepsilon_2 = \langle E_2, \alpha^{E_2'}\rangle$, $\varepsilon_3 = \langle \alpha^{E_3}, E_3'\rangle$, $\varepsilon_4 = \langle \alpha_4^{E}, E_4'\rangle$). By the definition of $\leqslant$, we know that $\langle E_1, E_1'\rangle \leqslant \langle E_2, E_2'\rangle$ and $\langle E_3, E_3'\rangle \leqslant \langle E_4, E_4'\rangle$. By the definition of transitivity we know that $\langle E_1, \alpha^{E_1'}\rangle \fatsemi \langle \alpha^{E_3}, E_3'\rangle = \langle E_5, E_5'\rangle$ and $\langle E_2, \alpha^{E_2'}\rangle \fatsemi \langle \alpha^{E_4}, E_4'\rangle = \langle E_6, E_6'\rangle$, where $\langle E_5, E_5'\rangle = \langle E_1, E_1'\rangle \fatsemi \langle E_3, E_3'\rangle$ and $\langle E_6, E_6'\rangle = \langle E_2, E_2'\rangle \fatsemi \langle E_4, E_4'\rangle$. Therefore, we are required to prove that $\langle E_5, E_5'\rangle \leqslant \langle E_6, E_6'\rangle$. But the result follows immediately by the induction hypothesis on $\langle E_1, E_1'\rangle \leqslant \langle E_2, E_2'\rangle$ and $\langle E_3, E_3'\rangle \leqslant \langle E_4, E_4'\rangle$.

Case ([α_3]- $\varepsilon_1 = \langle E_1, E_1'\rangle$, $\varepsilon_2 = \langle E_2, E_2'\rangle$, $\varepsilon_3 = \langle E_3, \alpha^{E_3'}\rangle$, $\varepsilon_4 = \langle E_4, \alpha^{E_4'}\rangle$). By the definition of $\leqslant$, we know that $\langle E_1, E_1'\rangle \leqslant \langle E_2, E_2'\rangle$ and $\langle E_3, E_3'\rangle \leqslant \langle E_4, E_4'\rangle$. By the definition of transitivity we know that $\langle E_1, E_1'\rangle \fatsemi \langle E_3, \alpha^{E_3'}\rangle = \langle E_5, \alpha^{E_5'}\rangle$ and $\langle E_2, E_2'\rangle \fatsemi \langle E_4, \alpha^{E_4'}\rangle = \langle E_6, \alpha^{E_6'}\rangle$, where $\langle E_5, E_5'\rangle = \langle E_1, E_1'\rangle \fatsemi \langle E_3, E_3'\rangle$ and $\langle E_6, E_6'\rangle = \langle E_2, E_2'\rangle \fatsemi \langle E_4, E_4'\rangle$. Therefore, we are required to prove that $\langle E_5, \alpha^{E_5'}\rangle \leqslant \langle E_6, \alpha^{E_6'}\rangle$, or what is the same $\langle E_5, E_5'\rangle \leqslant \langle E_6, E_6'\rangle$. But the result follows immediately by the induction hypothesis on $\langle E_1, E_1'\rangle \leqslant \langle E_2, E_2'\rangle$ and $\langle E_3, E_3'\rangle \leqslant \langle E_4, E_4'\rangle$.

Case ([∀]- $\varepsilon_i = \langle \forall X.E_i, \forall X.E'_i \rangle$). By the definition of $\leqslant$, we know that $\langle E_1, E'_1\rangle \leqslant \langle E_2, E'_2\rangle$ and $\langle E_3, E'_3\rangle \leqslant \langle E_4, E'_4\rangle$. By the definition of transitivity we know that $\langle \forall X.E_1, \forall X.E'_1\rangle \mathbin{;} \langle \forall X.E_3, \forall X.E'_3\rangle = \langle \forall X.E_5, \forall X.E'_5\rangle$ and $\langle \forall X.E_2, \forall X.E'_2\rangle \mathbin{;} \langle \forall X.E_4, \forall X.E'_4\rangle = \langle \forall X.E_6, \forall X.E'_6\rangle$, where $\langle E_5, E'_5\rangle = \langle E_1, E'_1\rangle \mathbin{;} \langle E_3, E'_3\rangle$ and $\langle E_6, E'_6\rangle = \langle E_2, E'_2\rangle \mathbin{;} \langle E_4, E'_4\rangle$. Therefore, we are required to prove that $\langle E_5, E'_5\rangle \leqslant \langle E_6, E'_6\rangle$. But the result follows immediately by the induction hypothesis on $\langle E_1, E'_1\rangle \leqslant \langle E_2, E'_2\rangle$ and $\langle E_3, E'_3\rangle \leqslant \langle E_4, E'_4\rangle$.

Case ([→]- $\varepsilon_i = \langle E_{1i} \to E_{2i}, E'_{1i} \to E'_{2i}\rangle$). By the definition of $\leqslant$, we know that $\langle E_{11}, E'_{11}\rangle \leqslant \langle E_{12}, E'_{12}\rangle$, $\langle E_{13}, E'_{13}\rangle \leqslant \langle E_{14}, E'_{14}\rangle$, $\langle E_{21}, E'_{21}\rangle \leqslant \langle E_{22}, E'_{22}\rangle$ and $\langle E_{23}, E'_{23}\rangle \leqslant \langle E_{24}, E'_{24}\rangle$. By the definition of transitivity we know that $\langle E_{11} \to E_{21}, E'_{11} \to E'_{21}\rangle \mathbin{;} \langle E_{13} \to E_{23}, E'_{13} \to E'_{23}\rangle = \langle E_{15} \to E_{25}, E'_{15} \to E'_{25}\rangle$ and $\langle E_{12} \to E_{22}, E'_{12} \to E'_{22}\rangle \mathbin{;} \langle E_{14} \to E_{24}, E'_{14} \to E'_{24}\rangle = \langle E_{16} \to E_{26}, E'_{16} \to E'_{26}\rangle$, where $\langle E'_{15}, E_{15}\rangle = \langle E'_{13}, E_{13}\rangle \mathbin{;} \langle E'_{11}, E_{11}\rangle$, $\langle E_{25}, E'_{25}\rangle = \langle E_{21}, E'_{21}\rangle \mathbin{;} \langle E_{24}, E'_{24}\rangle$, $\langle E_{16}, E'_{16}\rangle = \langle E'_{41}, E_{41}\rangle \mathbin{;} \langle E'_{12}, E_{12}\rangle$ and $\langle E_{26}, E'_{26}\rangle = \langle E_{22}, E'_{22}\rangle \mathbin{;} \langle E_{24}, E'_{24}\rangle$.

Therefore, we are required to prove that

$$\langle E_{15} \to E_{25}, E'_{15} \to E'_{25}\rangle \leqslant \langle E_{16} \to E_{26}, E'_{16} \to E'_{26}\rangle$$

or what is the same

$$\langle E'_{13}, E_{13}\rangle \mathbin{;} \langle E'_{11}, E_{11}\rangle = \langle E'_{15}, E_{15}\rangle \leqslant \langle E'_{16}, E_{16}\rangle = \langle E'_{41}, E_{41}\rangle \mathbin{;} \langle E'_{12}, E_{12}\rangle$$

and

$$\langle E_{21}, E'_{21}\rangle \mathbin{;} \langle E_{23}, E'_{23}\rangle = \langle E_{25}, E'_{25}\rangle \leqslant \langle E_{26}, E'_{26}\rangle = \langle E_{22}, E'_{22}\rangle \mathbin{;} \langle E_{24}, E'_{24}\rangle$$

But the result follows immediately by the induction hypothesis on$\langle E_{11}, E'_{11}\rangle \leqslant \langle E_{12}, E'_{12}\rangle$ and $\langle E_{13}, E'_{13}\rangle \leqslant \langle E_{14}, E'_{14}\rangle$, $\langle E_{21}, E'_{21}\rangle \leqslant \langle E_{22}, E'_{22}\rangle$ and $\langle E_{23}, E'_{23}\rangle \leqslant \langle E_{24}, E'_{24}\rangle$.

Case ([×]- $\varepsilon_i = \langle E_{1i} \times E_{2i}, E'_{1i} \times E'_{2i}\rangle$). Similar to Case [→].

Case ([$?_1$]- $\varepsilon_1 = \langle ?, ?\rangle$). Since $\varepsilon_1 \leqslant \varepsilon_2$, we know that $\varepsilon_2 = \langle ?, ?\rangle$. Therefore, by the transitivity rules, we know that $\varepsilon_1 \mathbin{;} \varepsilon_3 = \varepsilon_3$ and $\varepsilon_2 \mathbin{;} \varepsilon_4 = \varepsilon_4$. Thus, we are required to prove that $\varepsilon_3 \leqslant \varepsilon_4$, but the result follows immediately by premise.

Case ([$?_2$]- $\varepsilon_2 = \langle ?, ?\rangle$). The proof follows from some of the previous cases.

- ($\varepsilon_1 = \langle ?, ?\rangle$). The results follows immediately, since it was discussed in Case [$?_1$].
- ($\varepsilon_3 = \langle ?, ?\rangle$). The results follows immediately, since it was discussed in Case [$?_3$].
- ($\varepsilon_4 = \langle ?, ?\rangle$). The results follows immediately, since $\varepsilon_1 \mathbin{;} \varepsilon_3 \leqslant \langle ?, ?\rangle \mathbin{;} \langle ?, ?\rangle = \langle ?, ?\rangle$.
- ($\varepsilon_i = \langle B, B\rangle$). The results follows immediately, since $\langle B, B\rangle \mathbin{;} \langle B, B\rangle \leqslant \langle ?, ?\rangle \mathbin{;} \langle B, B\rangle$.
- ($\varepsilon_i = \langle X, X\rangle$). This case is not possible, since $\langle X, X\rangle \nleqslant \langle ?, ?\rangle$.
- Case [α_1] ($\varepsilon_1 = \langle E_1, \alpha^{E'_1}\rangle$, $\varepsilon_2 = \langle ?, ?\rangle$, $\varepsilon_3 = \langle \alpha^{E_3}, E'_3\rangle$, $\varepsilon_4 = \langle \alpha_4^{E}, E'_4\rangle$). This case is not possible, since $\langle \alpha^{E_1}, E'_1\rangle \nleqslant \langle ?, ?\rangle$.
- Case [α_2] ($\varepsilon_1 = \langle E_1, \alpha^{E'_1}\rangle$, $\varepsilon_2 = \langle ?, ?\rangle$, $\varepsilon_3 = \langle \alpha^{E_3}, E'_3\rangle$, $\varepsilon_4 = \langle \alpha_4^{E}, E'_4\rangle$). This case is not possible, since $\langle E_1, \alpha^{E'_1}\rangle \nleqslant \langle ?, ?\rangle$.
- Case [α_3] ($\varepsilon_1 = \langle E_1, E'_1\rangle$, $\varepsilon_2 = \langle ?, ?\rangle$, $\varepsilon_3 = \langle E_3, \alpha^{E'_3}\rangle$, $\varepsilon_4 = \langle E_4, \alpha^{E'_4}\rangle$). This case was discussed in Case [α_3] above.
- ($\varepsilon_1 = \langle \forall X.E_1, \forall X.E'_1\rangle$). This case is not possible, since $\langle \forall X.E_1, \forall X.E'_1\rangle \nleqslant \langle ?, ?\rangle$.
- ($\varepsilon_i = \langle E_{1i} \to E_{2i}, E'_{1i} \to E'_{2i}\rangle$). We have to prove that

  $$\langle E_{11} \to E_{21}, E'_{11} \to E'_{21}\rangle \mathbin{;} \langle E_{13} \to E_{23}, E'_{13} \to E'_{23}\rangle \leqslant \langle ?, ?\rangle \mathbin{;} \langle E_{14} \to E_{24}, E'_{14} \to E'_{24}\rangle$$

  or what is the same

  $$\langle E_{11} \to E_{21}, E'_{11} \to E'_{21}\rangle \mathbin{;} \langle E_{13} \to E_{23}, E'_{13} \to E'_{23}\rangle \leqslant \langle ? \to ?, ? \to ?\rangle \mathbin{;} \langle E_{14} \to E_{24}, E'_{14} \to E'_{24}\rangle$$

  But, this case was discussed in Case [→] above.

- ($\varepsilon_i = \langle E_{1i} \times E_{2i}, E'_{1i} \times E'_{2i} \rangle$). We have to prove that
$$\langle E_{11} \times E_{21}, E'_{11} \times E'_{21} \rangle \circ \langle E_{13} \times E_{23}, E'_{13} \times E'_{23} \rangle \leqslant \langle ?, ? \rangle \circ \langle E_{14} \times E_{24}, E'_{14} \times E'_{24} \rangle$$
or what is the same:
$$\langle E_{11} \times E_{21}, E'_{11} \times E'_{21} \rangle \circ \langle E_{13} \times E_{23}, E'_{13} \times E'_{23} \rangle \leqslant \langle ? \times ?, ? \times ? \rangle \circ \langle E_{14} \times E_{24}, E'_{14} \times E'_{24} \rangle$$
This case was discussed in Case [$\times$].

Case ([$?_3$]- $\varepsilon_3 = \langle ?, ? \rangle$). Since $\varepsilon_3 \leqslant \varepsilon_4$, we know that $\varepsilon_4 = \langle ?, ? \rangle$. Therefore, by the transitivity rules, we know that $\varepsilon_1 \circ \varepsilon_3 = \varepsilon_1$ and $\varepsilon_2 \circ \varepsilon_4 = \varepsilon_2$. Thus, we are required to prove that $\varepsilon_1 \leqslant \varepsilon_2$, but the result follows immediately by premise.

Case ([$?_4$]- $\varepsilon_4 = \langle ?, ? \rangle$). The proof follows from some of the previous cases.

- ($\varepsilon_1 = \langle ?, ? \rangle$). The results follows immediately, since it was discussed in Case [$?_1$].
- ($\varepsilon_2 = \langle ?, ? \rangle$). The results follows immediately, since $\varepsilon_1 \circ \varepsilon_3 \leqslant \langle ?, ? \rangle \circ \langle ?, ? \rangle = \langle ?, ? \rangle$.
- ($\varepsilon_3 = \langle ?, ? \rangle$). The results follows immediately, since it was discussed in Case [$?_3$].
- ($\varepsilon_i = \langle B, B \rangle$). The results follows immediately, since $\langle B, B \rangle \circ \langle B, B \rangle \leqslant \langle B, B \rangle \circ \langle ?, ? \rangle$.
- ($\varepsilon_i = \langle X, X \rangle$). This case is not possible, since $\langle X, X \rangle \nleqslant \langle ?, ? \rangle$.
- Case [α_1] ($\varepsilon_1 = \langle \alpha^{E_1}, E'_1 \rangle$, $\varepsilon_2 = \langle \alpha^{E_2}, E'_2 \rangle$, $\varepsilon_3 = \langle E_3, E'_3 \rangle$, $\varepsilon_4 = \langle ?, ? \rangle$). This case was discussed in Case [α_1] above.
- Case [α_2] ($\varepsilon_1 = \langle E_1, \alpha^{E'_1} \rangle$, $\varepsilon_2 = \langle E_2, \alpha^{E'_2} \rangle$, $\varepsilon_3 = \langle \alpha^{E_3}, E'_3 \rangle$, $\varepsilon_4 = \langle ?, ? \rangle$). This case is not possible, since $\langle E_3, \alpha^{E'_3} \rangle \nleqslant \langle ?, ? \rangle$.
- Case [α_3] ($\varepsilon_1 = \langle E_1, E'_1 \rangle$, $\varepsilon_2 = \langle E_2, E'_2 \rangle$, $\varepsilon_3 = \langle E_3, \alpha^{E'_3} \rangle$, $\varepsilon_4 = \langle ?, ? \rangle$). This case is not possible, since $\langle E_3, \alpha^{E'_3} \rangle \nleqslant \langle ?, ? \rangle$.
- ($\varepsilon_1 = \langle \forall X.E_1, \forall X.E'_1 \rangle$). This case is not possible, since $\langle \forall X.E_1, \forall X.E'_1 \rangle \nleqslant \langle ?, ? \rangle$.
- ($\varepsilon_i = \langle E_{1i} \rightarrow E_{2i}, E'_{1i} \rightarrow E'_{2i} \rangle$). We have to prove that
$$\langle E_{11} \rightarrow E_{21}, E'_{11} \rightarrow E'_{21} \rangle \circ \langle E_{13} \rightarrow E_{23}, E'_{13} \rightarrow E'_{23} \rangle \leqslant \langle E_{14} \rightarrow E_{24}, E'_{14} \rightarrow E'_{24} \rangle \circ \langle ?, ? \rangle$$
or what is the same
$$\langle E_{11} \rightarrow E_{21}, E'_{11} \rightarrow E'_{21} \rangle \circ \langle E_{13} \rightarrow E_{23}, E'_{13} \rightarrow E'_{23} \rangle \leqslant \langle E_{14} \rightarrow E_{24}, E'_{14} \rightarrow E'_{24} \rangle \circ \langle ? \rightarrow ?, ? \rightarrow ? \rangle$$
But, this case was discussed in Case [$\rightarrow$] above.
- ($\varepsilon_i = \langle E_{1i} \times E_{2i}, E'_{1i} \times E'_{2i} \rangle$). We have to prove that
$$\langle E_{11} \times E_{21}, E'_{11} \times E'_{21} \rangle \circ \langle E_{13} \times E_{23}, E'_{13} \times E'_{23} \rangle \leqslant \langle E_{14} \times E_{24}, E'_{14} \times E'_{24} \rangle \circ \langle ?, ? \rangle$$
or what is the same:
$$\langle E_{11} \times E_{21}, E'_{11} \times E'_{21} \rangle \circ \langle E_{13} \times E_{23}, E'_{13} \times E'_{23} \rangle \leqslant \langle E_{14} \times E_{24}, E'_{14} \times E'_{24} \rangle \circ \langle ? \times ?, ? \times ? \rangle$$
This case was discussed in Case [$\times$].

□

Definition 5.1 (Store Precision). $\Xi_1 \leqslant \Xi_2 \iff \Xi_1 = \Xi'_1, \alpha := G_1, \Xi_2 = \Xi'_2, \alpha := G_2, G_1 \leqslant G_2$ and $\Xi'_1 \leqslant \Xi'_2$, or $\Xi_1 = \Xi_2 = \cdot$.

Definition 5.2 (Typing Environment Precision). $\Gamma_1 \sqsubseteq \Gamma_2 \iff \Gamma_1 = \Gamma'_1, x : G_1$, $\Gamma_2 = \Gamma'_2, x : G_2$, $G_1 \leqslant G_2$ and $\Gamma'_1 \sqsubseteq \Gamma'_2$, or $\Gamma_1 = \Gamma_2 = \cdot$.

Proposition 5.3 (Lift Environment Precision). *If $G_1 \leqslant G_2$ and $\Xi_1 \leqslant \Xi_2$, then $\hat{G}_1 \leqslant \hat{G}_2$, where $\hat{G}_1 = lift_{\Xi_1}(G_1)$ and $\hat{G}_2 = lift_{\Xi_2}(G_2)$.*

Proof. Remember that

$$lift_{\Xi}(G) = \begin{cases} lift_{\Xi}(G_1) \rightarrow lift_{\Xi}(G_2) & G = G_1 \rightarrow G_2 \\ \forall X. lift_{\Xi}(G_1) & G = \forall X.G_1 \\ lift_{\Xi}(G_1) \times lift_{\Xi}(G_2) & G = G_1 \times G_2 \\ \alpha^{lift_{\Xi}(\Xi(\alpha))} & G = \alpha \\ G & \text{otherwise} \end{cases}$$

The prove follows by the definition of $\hat{G_1} = lift_{\Xi_1}(G_1)$ and induction on the structure of the type.

Case ($G_i = B$). The result follows immediately due to $\hat{B} = B \leqslant B = \hat{B}$.

Case ($G_i = X$). The result follows immediately due to $\hat{X} = X \leqslant X = \hat{X}$.

Case ($G_i = \alpha$). We are required to prove that $\alpha^{lift_{\Xi_1}(\Xi_1(\alpha))} \leqslant \alpha^{lift_{\Xi_2}(\Xi_2(\alpha))}$, or what is the same $lift_{\Xi_1}(\Xi_1(\alpha)) \leqslant lift_{\Xi_2}(\Xi_2(\alpha))$. Note that $\Xi_1(\alpha) \leqslant \Xi_2(\alpha)$ due to $\Xi_1 \leqslant \Xi_2$. The result follows immediately by the induction hypothesis on $\Xi_1(\alpha) \leqslant \Xi_2(\alpha)$ and $\Xi_1 \leqslant \Xi_2$.

Case ($G_i = \forall X.G_i'$). We know that $G_1' \leqslant G_2'$. We are required to prove that $\forall X.lift_{\Xi_1}(G_1') \leqslant \forall X.lift_{\Xi_2}(G_2')$, or what is the same $lift_{\Xi_1}(G_1') \leqslant lift_{\Xi_2}(G_2')$. By the induction hypothesis on $G_1' \leqslant G_2'$ and $\Xi_1 \leqslant \Xi_2$ the result follows immediately.

Case ($G_i = G_i' \rightarrow G_i''$). We know that $G_1' \leqslant G_2'$ and $G_1'' \leqslant G_2''$. We are required to prove that $lift_{\Xi_1}(G_1') \rightarrow lift_{\Xi_1}(G_1'') \leqslant lift_{\Xi_2}(G_2') \rightarrow lift_{\Xi_2}(G_2'')$, or what is the same $lift_{\Xi_1}(G_1') \leqslant lift_{\Xi_2}(G_2')$ and $lift_{\Xi_1}(G_1'') \leqslant lift_{\Xi_2}(G_2'')$. By the induction hypothesis on $G_1' \leqslant G_2'$ and $G_1'' \leqslant G_2''$ with $\Xi_1 \leqslant \Xi_2$ the result follows immediately.

Case ($G_i = G_i' \times G_i''$). This case is similar to the function case above.

Case ($G_1 = ?$). Then $G_2 = ?$. The result follows immediately due to $\hat{?} = ? \leqslant ? = \hat{?}$.

Case ($G_2 = ?$). Note that $\hat{G_2} = \hat{?} = ?$. Therefore, we are required to prove that $\hat{G_1} \leqslant ?$.

- Case ($G_1 = B$). The result follows immediately, $\hat{B} = B \leqslant ?$.
- Case ($G_1 = X$). This case is not possible due to $X \nleqslant ?$.
- Case ($G_1 = \alpha$). This case is not possible due to $\alpha \nleqslant ?$.
- Case ($G_1 = \forall X.G_1'$). This case is not possible due to $\forall X.G_1' \nleqslant ?$.
- Case ($G_1 = G_1' \rightarrow G_2'$). We are required to prove that $lift_{\Xi_1}(G_1') \rightarrow lift_{\Xi_2}(G_2') \leqslant ?$, or what is the same $lift_{\Xi_1}(G_1') \rightarrow lift_{\Xi_2}(G_2') \leqslant ? \rightarrow ?$, which follows similar to the function case above.
- Case ($G_1 = G_1' \times G_2'$). We are required to prove that $lift_{\Xi_1}(G_1') \times lift_{\Xi_2}(G_2') \leqslant ?$, or what is the same $lift_{\Xi_1}(G_1') \times lift_{\Xi_2}(G_2') \leqslant ? \times ?$, which follows similar to the pair case above.

□

Proposition 5.4 (Unlift Evidence Types Preserves Precision). *If $E_1 \leqslant E_2$ then $unlift(E_1) \leqslant unlift(E_2)$.*

Proof. Remember that

$$unlift(E) = \begin{cases} B & E = B \\ unlift(E_1) \to unlift(E_2) & E = E_1 \to E_2 \\ \forall X.unlift(E_1) & E = \forall X.E_1 \\ unlift(E_1) \times unlift(E_2) & E = E_1 \times E_2 \\ \alpha & E = \alpha^{E_1} \\ X & E = X \\ ? & E = ? \end{cases}$$

The prove follows by the definition of $unlift(E_1)$ and induction on the structure of the type.

Case ($G_i = B$). The result follows immediately due to $unlift(B) = B \leqslant B = unlift(B)$.

Case ($G_i = X$). The result follows immediately due to $unlift(X) = X \leqslant X = unlift(X)$.

Case ($G_i = \alpha^{E'_i}$). The result follows immediately due to $unlift(\alpha^{E'_1}) = \alpha \leqslant \alpha = unlift(\alpha^{E'_2})$.

Case ($E_i = \forall X.E'_i$). We know that $E'_1 \leqslant E'_2$. We are required to prove that $\forall X.unlift(E'_1) \leqslant \forall X.unlift(E'_2)$, or what is the same $unlift(E'_1) \leqslant unlift(E'_2)$. By the induction hypothesis on $E'_1 \leqslant E'_2$ the result follows immediately.

Case ($E_i = E'_i \to E''_i$). We know that $E'_1 \leqslant E'_2$ and $E''_1 \leqslant E''_2$. We are required to prove that $unlift(E'_1) \to unlift(E''_1) \leqslant unlift(E'_2) \to unlift(E''_2)$, or what is the same $unlift(E'_1) \leqslant unlift(E'_2)$ and $unlift(E''_1) \leqslant unlift(E''_2)$. By the induction hypothesis on $E'_1 \leqslant E'_2$ and $E''_1 \leqslant E''_2$ the result follows immediately.

Case ($E_i = E'_i \times E''_i$). This case is similar to the function case above.

Case ($E_1 = ?$). Then $E_2 = ?$. The result follows immediately due to $unlift(?) = ? \leqslant ? = unlift(?)$.

Case ($E_2 = ?$). Note that $unlift(E_2) = unlift(?) = ?$. Therefore, we are required to prove that $unlift(E_1) \leqslant ?$.

- Case ($E_1 = B$). The result follows immediately, $unlift(B) = B \leqslant ?$.
- Case ($E_1 = X$). This case is not possible due to $X \nleqslant ?$.
- Case ($E_1 = \alpha$). This case is not possible due to $\alpha \nleqslant ?$.
- Case ($E_1 = \forall X.E'_1$). This case is not possible due to $\forall X.E'_1 \nleqslant ?$.
- Case ($E_1 = E'_1 \to E'_2$). We are required to prove that $unlift(E'_1) \to unlift(E'_2) \leqslant ?$, or what is the same $unlift(E'_1) \to unlift(E'_2) \leqslant ? \to ?$, which follows similar to the function case above.
- Case ($E_1 = E'_1 \times E'_2$). We are required to prove that $unlift(E'_1) \times unlift(E'_2) \leqslant ?$, or what is the same $unlift(E'_1) \times unlift(E'_2) \leqslant ? \times ?$, which follows similar to the pair case above.

□

Proposition 5.5. *If* $\varepsilon_1 \leqslant \varepsilon_2$, $G_1 \leqslant G_2$, $\Xi_1 \leqslant \Xi_2$, $\alpha := G_1 \in \Xi_1$, $\alpha := G_2 \in \Xi_2$ *and* $\varepsilon_1[\hat{\alpha}_1/X]$ *is defined, then*

- $\varepsilon_1[\hat{\alpha}_1/X] \leqslant \varepsilon_2[\hat{\alpha}_2/X]$.
- $\langle E^*_1[\hat{\alpha}_1/X], E^*_1[\hat{G}_1/X]\rangle \leqslant \langle E^*_2[\hat{\alpha}_2/X], E^*_2[\hat{G}_2/X]\rangle$.

where $E^*_1 = lift_{\Xi_1}(unlift(\pi_2(\varepsilon_1)))$, $E^*_2 = lift_{\Xi_2}(unlift(\pi_2(\varepsilon_2)))$, $\hat{\alpha}_1 = lift_{\Xi_1}(\alpha_1)$, $\hat{\alpha}_2 = lift_{\Xi_2}(\alpha_2)$, $\hat{G}_1 = lift_{\Xi_1}(G_1)$ *and* $\hat{G}_2 = lift_{\Xi_2}(G_2)$.

Proof. Note that $\hat{\alpha}_1 \leqslant \hat{\alpha}_2$ and $\hat{G}_1 \leqslant \hat{G}_2$ by Proposition 5.3. Suppose that $\varepsilon_1 = \langle E, E' \rangle$ and $\varepsilon_2 = \langle E'', E''' \rangle$. We are required to prove that

$$\varepsilon_1[\hat{\alpha}_1/X] = \langle E[\hat{\alpha}_1/X], E'[\hat{\alpha}_1/X]\rangle \leqslant \langle E''[\hat{\alpha}_2/X], E'''[\hat{\alpha}_2/X]\rangle = \varepsilon_2[\hat{\alpha}_2/X]$$

$$\varepsilon_1{}^* = \langle E_1^*[\hat{\alpha}_1/X], E_1^*[\hat{G}_1/X]\rangle \leqslant \langle E_2^*[\hat{\alpha}_2/X], E_2^*[\hat{G}_2/X]\rangle = \varepsilon_2{}^*$$

We follow by case analysis on the evidence type, the definition of consistent transitivity for = and the definition of precision.

Case ($\varepsilon_i = \langle B, B\rangle$). The results follows immediately because $\varepsilon_1[\hat{\alpha}_1/X] = \varepsilon_2[\hat{\alpha}_2/X] = \varepsilon_1{}^* = \varepsilon_2{}^* = \langle B, B\rangle$.

Case ($\varepsilon_i = \langle X, X\rangle$). We are required to prove that $\varepsilon_1[\hat{\alpha}_1/X] = \langle \hat{\alpha}_1, \hat{\alpha}_1\rangle \leqslant \langle \hat{\alpha}_2, \hat{\alpha}_2\rangle = \varepsilon_2[\hat{\alpha}_2/X]$, which follows immediately due to $\hat{\alpha}_1 \leqslant \hat{\alpha}_2$. Also, we are required to prove that $\varepsilon_1{}^* = \langle \hat{\alpha}_1, \hat{G}_1\rangle \leqslant \langle \hat{\alpha}_2, \hat{G}_2\rangle = \varepsilon_2{}^*$, which follows immediately due to $\hat{\alpha}_1 \leqslant \hat{\alpha}_2$ and $\hat{G}_1 \leqslant \hat{G}_2$.

Case ($\varepsilon_i = \langle Y, Y\rangle$). The results follows immediately because $\varepsilon_1[\hat{\alpha}_1/X] = \varepsilon_2[\hat{\alpha}_2/X] = \varepsilon_1{}^* = \varepsilon_2{}^* = \langle Y, Y\rangle$.

Case ($\varepsilon_i = \langle \beta^{E_i}, E_i'\rangle$). The results follows immediately because $\varepsilon_1[\hat{\alpha}_1/X] = \langle \beta^{E_1}, E_1'\rangle \leqslant \langle \beta^{E_2}, E_2'\rangle = \varepsilon_2[\hat{\alpha}_2/X]$ by premise (note that X can not be free in $\langle \beta^{E_i}, E_i'\rangle$). Also, we are required to prove that $\varepsilon_1{}^* \leqslant \varepsilon_2{}^*$, but the result follows immediately by Preposition 5.4 and Proposition 5.3.

Case ($\varepsilon_i = \langle E_i, \beta^{E_i'}\rangle$). Similar to the previous case.

Case ($\varepsilon_i = \langle \forall Y.E_i, \forall Y.E_i'\rangle$). By the definition of $\leqslant$, we know that $\langle E_1, E_1'\rangle \leqslant \langle E_2, E_2'\rangle$. We are required to prove that

$$\varepsilon_1[\hat{\alpha}_1/X] = \langle \forall Y.E_1[\hat{\alpha}_1/X], \forall Y.E_1'[\hat{\alpha}_1/X]\rangle \leqslant \langle \forall Y.E_2[\hat{\alpha}_2/X], \forall Y.E_2'[\hat{\alpha}_2/X]\rangle = \varepsilon_2[\hat{\alpha}_2/X]$$

or what is the same

$$\langle E_1, E_1'\rangle\,[\hat{\alpha}_1/X] = \langle E_1[\hat{\alpha}_1/X], E_1'[\hat{\alpha}_1/X]\rangle \leqslant \langle E_2[\hat{\alpha}_2/X], E_2'[\hat{\alpha}_2/X]\rangle = \langle E_2, E_2'\rangle\,[\hat{\alpha}_2/X]$$

By the induction hypothesis on $\langle E_1, E_1'\rangle \leqslant \langle E_2, E_2'\rangle$ the result follows immediately.

Also we are required to prove

$$\varepsilon_1{}^* = \langle E_1^*[\hat{\alpha}_1/X], E_1^*[\hat{G}_1/X]\rangle \leqslant \langle E_2^*[\hat{\alpha}_2/X], E_2^*[\hat{G}_2/X]\rangle = \varepsilon_2{}^*$$

Note that $E_1^* = \mathit{lift}_{\Xi_1}(\mathit{unlift}(\forall Y.E_1')) = \forall Y.\mathit{lift}_{\Xi_1}(\mathit{unlift}(E_1')) = \forall Y.E_{11}^*$ and $E_2^* = \mathit{lift}_{\Xi_2}(\mathit{unlift}(\forall Y.E_2')) = \forall Y.\mathit{lift}_{\Xi_2}(\mathit{unlift}(E_2')) = \forall Y.E_{22}^*$. Therefore, we are required to prove

$$\langle E_{11}^*[\hat{\alpha}_1/X], E_{11}^*[\hat{G}_1/X]\rangle \leqslant \langle E_{22}^*[\hat{\alpha}_2/X], E_{22}^*[\hat{G}_2/X]\rangle$$

By the induction hypothesis on $\langle E_1, E_1'\rangle \leqslant \langle E_2, E_2'\rangle$ the result follows immediately.

Case ($\varepsilon_i = \langle E_{1i} \to E_{2i}, E_{1i}' \to E_{2i}'\rangle$). By the definition of $\leqslant$, we know that $\langle E_{11}, E_{11}'\rangle \leqslant \langle E_{12}, E_{12}'\rangle$ and $\langle E_{21}, E_{21}'\rangle \leqslant \langle E_{22}, E_{22}'\rangle$. We are required to prove that

$$\varepsilon_1[\hat{\alpha}_1/X] = \langle E_{11}[\hat{\alpha}_1/X] \to E_{12}[\hat{\alpha}_1/X], E_{11}'[\hat{\alpha}_1/X] \to E_{12}'[\hat{\alpha}_1/X]\rangle \leqslant$$
$$\langle E_{12}[\hat{\alpha}_2/X] \to E_{21}[\hat{\alpha}_2/X], E_{12}'[\hat{\alpha}_2/X] \to E_{22}'[\hat{\alpha}_2/X]\rangle = \varepsilon_2[\hat{\alpha}_2/X]$$

or what is the same

$$\langle E_{11}[\hat{\alpha}_1/X], E_{11}'[\hat{\alpha}_1/X]\rangle \leqslant \langle E_{12}[\hat{\alpha}_2/X], E_{12}'[\hat{\alpha}_2/X]\rangle$$

and

$$\langle E_{12}[\hat{\alpha}_1/X], E_{12}'[\hat{\alpha}_1/X]\rangle \leqslant \langle E_{21}[\hat{\alpha}_2/X], E_{22}'[\hat{\alpha}_2/X]\rangle$$

By the induction hypothesis on $\langle E_{11}, E_{11}'\rangle \leqslant \langle E_{12}, E_{12}'\rangle$ and $\langle E_{21}, E_{21}'\rangle \leqslant \langle E_{22}, E_{22}'\rangle$ the result follows immediately.

Also we are required to prove

$$\varepsilon_1{}^* = \langle E_1^*[\hat{\alpha}_1/X], E_1^*[\hat{G}_1/X]\rangle \leqslant \langle E_2^*[\hat{\alpha}_2/X], E_2^*[\hat{G}_2/X]\rangle = \varepsilon_2{}^*$$

Note that $E_1^* = \mathit{lift}_{\Xi_1}(\mathit{unlift}(E'_{11} \to E'_{12})) = \mathit{lift}_{\Xi_2}(\mathit{unlift}(E'_{11})) \to \mathit{lift}_{\Xi_2}(\mathit{unlift}(E'_{12})) = E_{11}^* \to E_{12}^*$ and $E_2^* = \mathit{lift}_{\Xi_2}(\mathit{unlift}(E'_{21} \to E'_{22})) = \mathit{lift}_{\Xi_2}(\mathit{unlift}(E'_{21})) \to \mathit{lift}_{\Xi_2}(\mathit{unlift}(E'_{22})) = E_{21}^* \to E_{22}^*$. Therefore, we are required to prove

$$\langle E_{11}^*[\hat{\alpha}_1/X], E_{11}^*[\hat{G}_1/X]\rangle \leqslant \langle E_{21}^*[\hat{\alpha}_2/X], E_{21}^*[\hat{G}_2/X]\rangle$$

and

$$\langle E_{12}^*[\hat{\alpha}_1/X], E_{12}^*[\hat{G}_1/X]\rangle \leqslant \langle E_{22}^*[\hat{\alpha}_2/X], E_{22}^*[\hat{G}_2/X]\rangle$$

By the induction hypothesis on $\langle E_{11}, E'_{11}\rangle \leqslant \langle E_{12}, E'_{12}\rangle$ and $\langle E_{21}, E'_{21}\rangle \leqslant \langle E_{22}, E'_{22}\rangle$ the result follows immediately.

Case ($\varepsilon_i = \langle E_{1i} \times E_{2i}, E'_{1i} \times E'_{2i}\rangle$). Similar to the function case.

Case ($\varepsilon_1 = \langle ?, ?\rangle$). Note that if $\varepsilon_1 = \langle ?, ?\rangle$ then $\varepsilon_2 = \langle ?, ?\rangle$. Therefore, the result follows immediately because $\varepsilon_1[\hat{\alpha}_1] = \varepsilon_2[\hat{\alpha}_2] = \varepsilon_1{}^* = \varepsilon_2{}^* = \langle ?, ?\rangle$. This case is trivial,

Case ($\varepsilon_2 = \langle ?, ?\rangle$). Note that $\varepsilon_2[\hat{\alpha}_2] = \varepsilon_2{}^* = \langle ?, ?\rangle$. Therefore, we are required to prove that $\varepsilon_1[\hat{\alpha}_1] \leqslant \langle ?, ?\rangle$ and $\varepsilon_1{}^* \leqslant \langle ?, ?\rangle$.

- Case ($\varepsilon_1 = \langle B, B\rangle$). The result follows immediately, $\varepsilon_1[\hat{\alpha}_1/X] = \varepsilon_1{}^* = \langle B, B\rangle \leqslant \langle ?, ?\rangle$.
- Case ($\varepsilon_1 = \langle X, X\rangle$). This case is not possible due to $\langle X, X\rangle \nleqslant \langle ?, ?\rangle$.
- Case ($\varepsilon_1 = \langle \alpha^{E_1}, E'_1\rangle$). This case is not possible due to $\langle \alpha^{E_1}, E'_1\rangle \nleqslant \langle ?, ?\rangle$.
- Case ($\varepsilon_1 = \langle E_1, \alpha^{E'_1}\rangle$). This case is not possible due to $\langle E_1, \alpha^{E'_1}\rangle \nleqslant \langle ?, ?\rangle$.
- Case ($\varepsilon_1 = \langle \forall Y.E_1, \forall Y.E'_1\rangle$). This case is not possible due to $\langle \forall Y.E_1, \forall Y.E'_1\rangle \nleqslant \langle ?, ?\rangle$.
- Case ($\varepsilon_1 = \langle E_{11} \to E_{12}, E'_{11} \to E'_{12}\rangle$). We are required to prove that $\varepsilon_1[\hat{\alpha}_1] \leqslant \langle ?, ?\rangle$ and $\varepsilon_1{}^* \leqslant \langle ?, ?\rangle$, or what is the same $\varepsilon_1[\hat{\alpha}_1] \leqslant \langle ? \to ?, ? \to ?\rangle$ and $\varepsilon_1{}^* \leqslant \langle ? \to ?, ? \to ?\rangle$, which follows similar to the function case above.
- Case ($\varepsilon_1 = \langle E_{11} \times E_{12}, E'_{11} \times E'_{12}\rangle$). We are required to prove that $\varepsilon_1[\hat{\alpha}_1] \leqslant \langle ?, ?\rangle$ and $\varepsilon_1{}^* \leqslant \langle ?, ?\rangle$, or what is the same $\varepsilon_1[\hat{\alpha}_1] \leqslant \langle ? \times ?, ? \times ?\rangle$ and $\varepsilon_1{}^* \leqslant \langle ? \times ?, ? \times ?\rangle$, which follows similar to the pair case above.

□

Proposition 5.6. *If $\varepsilon_1 \sqsubseteq \varepsilon_2$, $G_1 \leqslant G_2$, $\Xi_1 \leqslant \Xi_2$, $\alpha := G_1 \in \Xi_1$, $\alpha := G_2 \in \Xi_2$ and $\varepsilon_1[\hat{\alpha}_1/X]$ is defined, then $\varepsilon_1[\hat{\alpha}_1/X] \sqsubseteq \varepsilon_2[\hat{\alpha}_2/X]$, where $\hat{\alpha}_1 = \mathit{lift}_{\Xi_1}(\alpha)$ and $\hat{\alpha}_2 = \mathit{lift}_{\Xi_2}(\alpha)$.*

Proof. Similar to Proposition 5.5. □

Proposition 5.7 (Monotonicity of Evidence Instantiation). *If $\varepsilon_1 \leqslant \varepsilon_2$, $G_1 \leqslant G_2$, $\Xi_1 \leqslant \Xi_2$, $\alpha := G_1 \in \Xi_1$, $\alpha := G_2 \in \Xi_2$ and $\varepsilon_1[\hat{\alpha}_1]$ is defined, then*

- $\hat{\alpha}_1 \leqslant \hat{\alpha}_2$.
- $\varepsilon_1[\hat{\alpha}_1] \leqslant \varepsilon_2[\hat{\alpha}_2]$.
- $\varepsilon_{1out} \leqslant \varepsilon_{2out}$.

where $\hat{\alpha}_1 = \mathit{lift}_{\Xi_1}(\alpha)$ and $\hat{\alpha}_2 = \mathit{lift}_{\Xi_2}(\alpha)$.

Proof. This result $\hat{\alpha}_1 \leqslant \hat{\alpha}_2$ follows immediately by the Proposition 5.3.
Remember that

$$\varepsilon_{out} \triangleq \langle E_*[\alpha^E], E_*[E']\rangle \qquad \text{where } E_* = \mathit{lift}_\Xi(\mathit{unlift}(\pi_2(\varepsilon))), \alpha^E = \mathit{lift}_{\Xi'}(\alpha), E' = \mathit{lift}_\Xi(G')$$

Note that $\varepsilon_1[\hat{\alpha}_1]$ only succeed if $\varepsilon_1 = \langle \forall X.E, \forall X.E' \rangle$. Since $\varepsilon_1 \leqslant \varepsilon_2$ and $\varepsilon_1 = \langle \forall X.E, \forall X.E' \rangle$, then $\varepsilon_2 = \langle \forall X.E'', \forall X.E''' \rangle$. Let suppose that $\varepsilon_1' = \langle E, E' \rangle$ and $\varepsilon_2' = \langle E'', E''' \rangle$. Then we are required to prove that

$$\varepsilon_1[\hat{\alpha}_1] = \varepsilon_1'[\hat{\alpha}_1/X] = \langle E[\hat{\alpha}_1/X], E'[\hat{\alpha}_1/X] \rangle \leqslant \langle E''[\hat{\alpha}_2/X], E'''[\hat{\alpha}_2/X] \rangle = \varepsilon_2'[\hat{\alpha}_2/X] = \varepsilon_2[\hat{\alpha}_2]$$

$$\varepsilon_{1out} = \langle E_1^*[\hat{\alpha}_1/X], E_1^*[\hat{G}_1/X] \rangle \leqslant \langle E_2^*[\hat{\alpha}_2/X], E_2^*[\hat{G}_2/X] \rangle = \varepsilon_{2out}$$

where $E_1^* = \mathit{lift}_{\Xi_1}(\mathit{unlift}(E'))$, $E_2^* = \mathit{lift}_{\Xi_2}(\mathit{unlift}(E'''))$, $\hat{G}_1 = \mathit{lift}_{\Xi_1}(G_1)$ and $\hat{G}_2 = \mathit{lift}_{\Xi_2}(G_2)$. By the Proposition 5.5 the result follows immediately. □

Proposition 5.8. *If $G_1^* \sqsubseteq G_2^*$ and $G_1' \sqsubseteq G_2'$ then $G_1^*[G_1'/X] \sqsubseteq G_2^*[G_2'/X]$.*

Proof. Follow by induction on $G_1^* \sqsubseteq G_2^*$.

Case ($B \sqsubseteq B$). The results follows immediately due to $B[G_1'/X] = B \sqsubseteq B = B[G_2'/X]$.

Case ($Y \sqsubseteq Y$). If $Y = X$, the results follows immediately due to $X[G_1'/X] = G_1' \sqsubseteq G_2' = X[G_2'/X]$ and $G_1' \sqsubseteq G_2'$ by premise. If $Y \neq X$, the results, also, follows immediately due to $Y[G_1'/X] = Y \sqsubseteq Y = Y[G_2'/X]$.

Case ($\alpha \sqsubseteq \alpha$). The results follows immediately due to $\alpha[G_1'/X] = \alpha \sqsubseteq \alpha = \alpha[G_2'/X]$.

Case ($G \sqsubseteq ?$). The results follows immediately due to $G[G_1'/X] \sqsubseteq ? = ?[G_2'/X]$.

Case ($\forall X.G_1 \sqsubseteq \forall X.G_2$). We know that

$$\frac{G_1 \sqsubseteq G_2}{\forall X.G_1 \sqsubseteq \forall X.G_2}$$

By the definition of $\sqsubseteq$, we know that $G_1 \sqsubseteq G_2$. We are required to prove that

$$(\forall X.G_1)[G_1'/X] = (\forall X.G_1[G_1'/X]) \sqsubseteq (\forall X.G_2[G_2'/X]) = (\forall X.G_2)[G_2'/X]$$

Or what is the same that $(G_1[G_1'/X]) \sqsubseteq (G_2[G_2'/X])$. But the result follows immediately by the induction hypothesis on $G_1 \sqsubseteq G_2$.

Case ($G_1 \rightarrow G_2 \sqsubseteq G_3 \rightarrow G_4$). We know that

$$\frac{G_1 \sqsubseteq G_3 \qquad G_2 \sqsubseteq G_4}{G_1 \rightarrow G_2 \sqsubseteq G_3 \rightarrow G_4}$$

By the definition of $\sqsubseteq$, we know that $G_1 \sqsubseteq G_3$ and $G_2 \sqsubseteq G_4$. We are required to prove that

$$(G_1 \rightarrow G_2)[G_1'/X] = (G_1[G_1'/X] \rightarrow G_2[G_1'/X]) \sqsubseteq (G_3[G_2'/X] \rightarrow G_4[G_2'/X]) = (G_3 \rightarrow G_4)[G_2'/X]$$

Or what is the same that $G_1[G_1'/X] \sqsubseteq G_3[G_2'/X]$ and $G_2[G_1'/X] \sqsubseteq G_4[G_2'/X]$. But the result follows immediately by the induction hypothesis on $G_1 \sqsubseteq G_3$ and $G_2 \sqsubseteq G_4$.

Case ($G_1 \times G_2 \sqsubseteq G_3 \times G_4$). We know that

$$\frac{G_1 \sqsubseteq G_3 \qquad G_2 \sqsubseteq G_4}{G_1 \times G_2 \sqsubseteq G_3 \times G_4}$$

By the definition of $\sqsubseteq$, we know that $G_1 \sqsubseteq G_3$ and $G_2 \sqsubseteq G_4$. We are required to prove that

$$(G_1 \times G_2)[G_1'/X] = (G_1[G_1'/X] \times G_2[G_1'/X]) \sqsubseteq (G_3[G_2'/X] \times G_4[G_2'/X]) = (G_3 \times G_4)[G_2'/X]$$

Or what is the same that $G_1[G_1'/X] \sqsubseteq G_3[G_2'/X]$ and $G_2[G_1'/X] \sqsubseteq G_4[G_2'/X]$. But the result follows immediately by the induction hypothesis on $G_1 \sqsubseteq G_3$ and $G_2 \sqsubseteq G_4$.

□

Proposition 5.9. *If* $G_1 \sqsubseteq G_2$ *and* $G_1' \leqslant G_2'$ *then* $G_1[G_1'/X] \sqsubseteq G_2[G_2'/X]$.

Proof. By Proposition 5.14 and Proposition 5.8 the results follows immediately. □

Proposition 5.10. *If* $G_1 \leqslant G_2$ *and* $G_1' \leqslant G_2'$ *then* $G_1[G_1'/X] \leqslant G_2[G_2'/X]$.

Proof. Straightforward induction on $G_1 \leqslant G_2$. Very similar to Proposition 5.8. □

Proposition 5.11. *If* $G_1 \rightarrow\!\!\triangleright\, G_2$ *then* $G_1[\alpha/X] \rightarrow\!\!\triangleright\, G_2[\alpha/X]$.

Proof. By induction on the definition of $G_1 \rightarrow\!\!\triangleright\, G_2$. □

5.3 Weak Dynamic Gradual Guarantee for GSF

In this section, we present the proof of the weak dynamic gradual guarantee for GSFε previously presented and the auxiliary Propositions an Definitions.

Proposition 5.12 (Monotonicity of Evidence Substitution). *If* $\Omega \vdash s_1^* \leqslant s_2^* : G_1^* \leqslant G_2^*$ *and* $\Xi_1 \leqslant \Xi_2$, *then* $\Omega[\alpha/X] \vdash s_1^*[\hat{\alpha_1}/X] \leqslant s_2^*[\hat{\alpha_2}/X] : G_1^*[\alpha/X] \leqslant G_2^*[\alpha/X]$, *where* $\alpha := G_1^{**} \in \Xi_1$, $\alpha := G_2^{**} \in \Xi_2$, $\hat{\alpha_1} = lift_{\Xi_1}(\alpha)$ *and* $\hat{\alpha_2} = lift_{\Xi_2}(\alpha)$.

Proof. We follow by induction on $\Omega \vdash s_1^* \leqslant s_2^* : G_1^* \leqslant G_2^*$. We avoid the notation $\Omega \vdash s_1^* \leqslant s_2^* : G_1^*[\alpha/X] \leqslant G_2^*[\alpha/X]$, and use $s_1^* \leqslant s_2^*$ instead, for simplicity, when the typing environments are not relevant.

Case ($b \leqslant b$). The results follows immediately due to $b[\hat{\alpha_1}/X] = b \leqslant b = b[\hat{\alpha_2}/X]$.

Case ($x \leqslant x$). The results follows immediately due to $x[\hat{\alpha_1}/X] = x \leqslant x = x[\hat{\alpha_2}/X]$.

Case ($(\lambda x : G_1.t_1) \leqslant (\lambda x : G_2.t_2)$). We know that

$$\frac{\Omega \vdash \Xi_1 \triangleright t_1 : G_1' \leqslant \Xi_2 \triangleright t_2 : G_2' \qquad G_1 \sqsubseteq G_2}{(\lambda x : G_1.t_1) \leqslant (\lambda x : G_2.t_2)}$$

We are required to show

$$(\lambda x : G_1.t_1)[\hat{\alpha_1}/X] = (\lambda x : G_1[\alpha/X].t_1[\hat{\alpha_1}/X]) \leqslant (\lambda x : G_2[\alpha/X].t_2[\hat{\alpha_2}/X]) = (\lambda x : G_2.t_2)[\hat{\alpha_2}/X]$$

Note that $G_1[\alpha/X] \sqsubseteq G_2[\alpha/X]$, by Proposition 5.9.
Therefore, we are required to prove

$$\Omega, x : G_1[\alpha/X] \sqsubseteq G_2[\alpha/X] \vdash \Xi_1 \triangleright (t_1[\hat{\alpha_1}/X]) : G_1'[\alpha/X] \leqslant \Xi_2 \triangleright (t_2[\hat{\alpha_2}/X]) : G_2'[\alpha/X]$$

But the results follows immediately by the induction hypothesis on

$$\Omega, x : G_1 \sqsubseteq G_2 \vdash \Xi_1 \triangleright t_1 : G_1' \leqslant \Xi_2 \triangleright t_2 : G_2'$$

Case ($(\Lambda Y.t_1) \leqslant (\Lambda Y.t_2)$). We know that

$$\frac{t_1 \leqslant t_2}{(\Lambda Y.t_1) \leqslant (\Lambda Y.t_2)}$$

We are required to show

$$(\Lambda Y.t_1)[\hat{\alpha_1}/X] = (\Lambda Y.t_1[\hat{\alpha_1}/X]) \leqslant (\Lambda Y.t_2[\hat{\alpha_2}/X]) = (\Lambda Y.t_2)[\hat{\alpha_2}/X]$$

Therefore, we are required to prove $(t_1[\hat{\alpha_1}/X]) \leqslant (t_2[\hat{\alpha_2}/X])$. But the results follows immediately by the induction hypothesis on $t_1 \leqslant t_2$.

Case $(t_1\ t_2 \leqslant t_1\ t_2')$. We know that

$$\frac{t_1 \leqslant t_1' \qquad t_2 \leqslant t_2'}{t_1\ t_2 \leqslant t_1\ t_2'}$$

We are required to show

$$(t_1\ t_2)[\hat{\alpha_1}/X] = t_1[\hat{\alpha_1}/X]\ t_2[\hat{\alpha_1}/X]) \leqslant (t_1'[\hat{\alpha_2}/X]\ t_2'[\hat{\alpha_2}/X]) = (t_1'\ t_2')[\hat{\alpha_2}/X]$$

Therefore, we are required to prove $t_1[\hat{\alpha_1}/X] \leqslant t_1'[\hat{\alpha_2}/X]$ and $t_2[\hat{\alpha_1}/X] \leqslant t_2'[\hat{\alpha_2}/X]$. But the results follows immediately by the induction hypothesis on $t_1 \leqslant t_1'$ and $t_2 \leqslant t_2'$.

Case $(t_1\ [G_1] \leqslant t_2\ [G_2])$. We know that

$$\frac{t_1 \leqslant t_2 \qquad G_1 \leqslant G_2}{t_1\ [G_1] \leqslant t_2\ [G_2]}$$

We are required to show

$$(t_1\ [G_1])[\hat{\alpha_1}/X] = (t_1[\hat{\alpha_1}/X]\ [G_1[\alpha/X]]) \leqslant (t_2[\hat{\alpha_2}/X]\ [G_2[\alpha/X]]) = (t_2\ [G_2])[\hat{\alpha_2}/X]$$

Note that $G_1[\alpha/X] \leqslant G_2[\alpha/X]$ by Proposition 5.10 and $G_1 \leqslant G_2$.
Therefore, we are required to prove $(t_1[\hat{\alpha_1}/X]) \leqslant (t_2[\hat{\alpha_2}/X])$. But the results follows immediately by the induction hypothesis on $t_1 \leqslant t_2$.

Case $(\varepsilon_1 s_1 :: G_1 \leqslant \varepsilon_2 s_2 :: G_2)$.

$$\frac{\varepsilon_1 \leqslant \varepsilon_2 \qquad s_1 \leqslant s_2 \qquad G_1 \sqsubseteq G_2}{\varepsilon_1 s_1 :: G_1 \leqslant \varepsilon_2 s_2 :: G_2}$$

We are required to show

$$(\varepsilon_1 s_1 :: G_1)[\hat{\alpha_1}/X] = (\varepsilon_1[\hat{\alpha_1}/X] s_1[\hat{\alpha_1}/X] :: G_1[\alpha/X]) \leqslant (\varepsilon_2[\hat{\alpha_2}/X] s_2[\hat{\alpha_2}/X] :: G_2[\alpha/X]) = (\varepsilon_2 s_2 :: G_2)[\hat{\alpha_2}/X]$$

Note that by Proposition 5.5 and $\varepsilon_1 \leqslant \varepsilon_2$, we know that $\varepsilon_1[\hat{\alpha_1}/X] \leqslant \varepsilon_2[\hat{\alpha_2}/X]$. Also, by Proposition 5.9 and $G_1 \sqsubseteq G_2$, we know that $G_1[\alpha/X] \sqsubseteq G_2[\alpha/X]$.
Therefore, we are required to prove $(s_1[\hat{\alpha_1}/X]) \leqslant (s_2[\hat{\alpha_2}/X])$. But the results follows immediately by the induction hypothesis on $s_1 \leqslant s_2$.

Case $(\varepsilon_{G_1} t_1' :: G_1 \leqslant \varepsilon_{G_2} t_2' :: G_2)$.

$$\frac{\Omega \vdash \Xi_1 \triangleright t_1' : G_1' \leqslant \Xi_2 \triangleright t_2' : G_2' \qquad G_1 \sqsubseteq G_2 \qquad G_1' \rightharpoonup G_1 \qquad G_2' \rightharpoonup G_2}{\Omega \vdash \Xi_1 \triangleright \varepsilon_{G_1}(t_1' :: G_1 : G_1 \leqslant \Xi_2 \triangleright \varepsilon_{G_2} t_2' :: G_2 : G_2}$$

We are required to show

$$(\varepsilon_{G_1} t_1' :: G_1)[\hat{\alpha_1}/X] = (\varepsilon_{G_1}[\hat{\alpha_1}/X] t_1'[\hat{\alpha_1}/X] :: G_1[\alpha/X]) \leqslant$$
$$(\varepsilon_{G_2}[\hat{\alpha_2}/X] t_2'[\hat{\alpha_2}/X] :: G_2[\alpha/X]) = (\varepsilon_{G_2} t_2' :: G_2)[\hat{\alpha_2}/X]$$

Note that since $G_1 \sqsubseteq G_2$ and Proposition 5.24, we know that $\varepsilon_{G_1} \sqsubseteq \varepsilon_{G_2}$. Note that by Proposition 5.6 and $\varepsilon_{G_1} \sqsubseteq \varepsilon_{G_2}$, we know that $\varepsilon_{G_1}[\hat{\alpha_1}/X] \sqsubseteq \varepsilon_{G_2}[\hat{\alpha_2}/X]$. Also, by Proposition 5.9 and $G_1 \sqsubseteq G_2$, we know that $G_1[\alpha/X] \sqsubseteq G_2[\alpha/X]$. By Proposition 5.11, we know that $G_1'[\alpha/X] \rightharpoonup G_1[\alpha/X]$ and $G_2'[\alpha/X] \rightharpoonup G_2[\alpha/X]$. Therefore, we are required to prove $(t_1[\hat{\alpha_1}/X]) \leqslant (t_2[\hat{\alpha_2}/X])$. But the results follows immediately by the induction hypothesis on $t_1 \leqslant t_2$.

□

Proposition 5.13 (Substitution Preserves Precision). *If $\Omega', x : G_1 \sqsubseteq G_2 \vdash s_1 \leqslant s_2 : G_1' \leqslant G_2'$ and $\Omega' \vdash v_1 \leqslant v_2 : G_1 \leqslant G_2$, then $\Omega' \vdash s_1[v_1/x] \leqslant s_2[v_2/x] : G_1' \leqslant G_2'$.*

Proof. We follow by induction on $\Omega', x : G_1 \sqsubseteq G_2 \vdash t_1 \leqslant t_2 : G'_1 \leqslant G'_2$. We avoid the notation $\Omega', x : G_1 \sqsubseteq G_2 \vdash t_1 \leqslant t_2 : G'_1 \leqslant G'_2$, and use $t_1 \leqslant t_2$ instead, for simplicity, when the typing environments are not relevant. Let suppose that $\Omega = \Omega', x : G_1 \sqsubseteq G_2$.

Case $(b \leqslant b)$. The result follows immediately.

Case $(x \leqslant x)$. We know that

$$(\leqslant\mathrm{x}_\varepsilon)\frac{x : G_1 \sqsubseteq G_2 \in \Omega}{\Omega \vdash \Xi_1 \triangleright x : G_1 \leqslant \Xi_2 \triangleright x : G_2}$$

The result follows immediately due to $\Omega \vdash \Xi_1 \triangleright v_1 : G_1 \leqslant \Xi_2 \triangleright v_2 : G_2$ and

$$t_1[v_1/x] = x[v_1/x] = v_1 \leqslant v_2 = x[v_2/x] = t_2[v_2/x]$$

Case $((\lambda y : G''_1.t'_1) \leqslant (\lambda y : G''_2.t'_2))$. We know that

$$\frac{\Omega, y : G''_1 \sqsubseteq G''_2 \vdash \Xi_1 \triangleright t'_1 : G'''_1 \leqslant \Xi_2 \triangleright t'_2 : G'''_2 \qquad G''_1 \sqsubseteq G''_2}{\Omega \vdash \Xi_1 \triangleright (\lambda y : G''_1.t'_1) : G''_1 \to G'''_1 \leqslant \Xi_2 \triangleright (\lambda y : G''_2.t'_2) : G''_2 \to G'''_2}$$

Note that we are required to prove that $\Omega \vdash \Xi_1 \triangleright (\lambda y : G''_1.t'_1) : G''_1 \to G'''_1 \leqslant \Xi_2 \triangleright (\lambda y : G''_2.t'_2) : G''_2 \to G'''_2$.

$$(\lambda y : G''_1.t'_1)[v_1/x] = (\lambda y : G''_1.t'_1[v_1/x]) \leqslant (\lambda y : G''_2.t'_2[v_2/x]) = (\lambda y : G''_2.t'_2)[v_2/x]$$

or what is the same $\Omega, y : G''_1 \sqsubseteq G''_2 \vdash \Xi_1 \triangleright t'_1[v_1/x] : G'''_1 \leqslant \Xi_2 \triangleright t'_2[v_2/x] : G'''_2$. But the result follows immediately by the induction hypothesis on $\Omega, y : G''_1 \sqsubseteq G''_2 \vdash \Xi_1 \triangleright t'_1 : G'''_1 \leqslant \Xi_2 \triangleright t'_2 : G'''_2$.

Case $((\Lambda X.t'_1) \leqslant (\Lambda X.t'_2))$. We know that

$$\frac{\Omega \vdash \Xi_1 \triangleright t'_1 : G''_1 \leqslant \Xi_2 \triangleright t'_2 : G''_2}{\Omega \vdash \Xi_1 \triangleright (\Lambda X.t'_1) : \forall X.G''_1 \leqslant \Xi_2 \triangleright (\Lambda X.t'_2) : \forall X.G''_2}$$

Note that we are required to prove that $\Omega \vdash \Xi_1 \triangleright (\Lambda X.t'_1) : \forall X.G''_1 \leqslant \Xi_2 \triangleright (\Lambda X.t'_2) : \forall X.G''_2$.

$$(\Lambda X.t'_1)[v_1/x] = (\Lambda X.t'_1[v_1/x]) \leqslant (\Lambda X.t'_2[v_2/x]) = (\Lambda X.t'_2)[v_2/x]$$

or what is the same $\Omega \vdash \Xi_1 \triangleright t'_1[v_1/x] : G''_1 \leqslant \Xi_2 \triangleright t'_2[v_2/x] : G''_2$. But the result follows immediately by the induction hypothesis on $\Omega \vdash \Xi_1 \triangleright t'_1 : G''_1 \leqslant \Xi_2 \triangleright t'_2 : G''_2$.

Case $(t'_1\ t'_2 \leqslant t''_1\ t''_2)$. We know that

$$\frac{\Omega \vdash \Xi_1 \triangleright t'_1 : G''_1 \to G'''_1 \leqslant \Xi_2 \triangleright t'_2 : G''_2 \to G'''_2 \qquad \Omega \vdash \Xi_1 \triangleright t''_1 : G''_1 \leqslant \Xi_2 \triangleright t''_2 : G''_2}{\Omega \vdash \Xi_1 \triangleright t'_1\ t''_1 : G'''_1 \leqslant \Xi_2 \triangleright t'_2\ t''_2 : G'''_2}$$

Note that we are required to prove that $\Omega \vdash \Xi_1 \triangleright t'_1\ t''_1 : G'''_1 \leqslant \Xi_2 \triangleright t'_2\ t''_2 : G'''_2$.

$$(t'_1\ t''_1)[v_1/x] = t'_1[v_1/x]\ t''_1[v_1/x] \leqslant t'_2[v_2/x]\ t''_2[v_2/x] = (t'_2\ t''_2)[v_2/x]$$

or what is the same $\Omega \vdash \Xi_1 \triangleright t'_1[v_1/x] : G''_1 \to G'''_1 \leqslant \Xi_2 \triangleright t'_2[v_2/x] : G''_2 \to G'''_2$ and $\Omega \vdash \Xi_1 \triangleright t''_1[v_1/x] : G'''_1 \leqslant \Xi_2 \triangleright t''_2[v_2/x] : G'''_2$. But the result follows immediately by the induction hypothesis on $\Omega \vdash \Xi_1 \triangleright t'_1 : G''_1 \to G'''_1 \leqslant \Xi_2 \triangleright t'_2 : G''_2 \to G'''_2$ and $\Omega \vdash \Xi_1 \triangleright t''_1 : G''_1 \leqslant \Xi_2 \triangleright t''_2 : G''_2$.

Case $(t'_1\ [G''_1] \leqslant t'_2\ [G''_2])$.

$$\frac{\Omega \vdash \Xi_1 \triangleright t'_1 : \forall X.G'''_1 \leqslant \Xi_2 \triangleright t'_2 : \forall X.G'''_2 \qquad G''_1 \leqslant G''_2}{\Omega \vdash \Xi_1 \triangleright t'_1\ [G''_1] : G'''_1[G''_1/X] \leqslant \Xi_2 \triangleright t'_2\ [G''_2] : G'''_2[G''_2/X]}$$

Note that we are required to prove that $\Omega \vdash \Xi_1 \triangleright t'_1\ [G''_1] : G'''_1[G''_1/X] \leqslant \Xi_2 \triangleright t'_2\ [G''_2] : G'''_2[G''_2/X]$.

$$(t'_1\ [G''_1])[v_1/x] = (t'_1[v_1/x]\ [G''_1]) \leqslant (t'_2[v_2/x]\ [G''_2]) = (t'_2\ [G''_2])[v_2/x]$$

or what is the same $\Omega \vdash \Xi_1 \triangleright t_1'[v_1/x] : G_1'''[G_1''/X] \leqslant \Xi_2 \triangleright t_2'[v_2/x] : G_2'''[G_2''/X]$. But the result follows immediately by the induction hypothesis on $\Omega \vdash \Xi_1 \triangleright t_1' : G_1'''[G_1''/X] \leqslant \Xi_2 \triangleright t_2' : G_2'''[G_2''/X]$.

Case $(\varepsilon_1 s_1' :: G_1'' \leqslant \varepsilon_1 s_1' :: G_1'')$.

$$\frac{\varepsilon_1 \leqslant \varepsilon_2 \qquad \Omega \vdash \Xi_1 \triangleright s_1' : G_1''' \leqslant \Xi_2 \triangleright s_2' : G_2''' \qquad G_1'' \sqsubseteq G_2''}{\Omega \vdash \Xi_1 \triangleright \varepsilon_1 s_1' :: G_1'' : G_1'' \leqslant \Xi_2 \triangleright \varepsilon_2 s_2' :: G_2'' : G_2''}$$

Note that we are required to prove that $\Omega \vdash \Xi_1 \triangleright \varepsilon_1 s_1' :: G_1'' : G_1'' \leqslant \Xi_2 \triangleright \varepsilon_2 s_2' :: G_2'' : G_2''$.

$$(\varepsilon_1 s_1' :: G_1'')[v_1/x] = (\varepsilon_1 s_1'[v_1/x] :: G_1'') \leqslant (\varepsilon_2 s_2'[v_2/x] :: G_2'') = (\varepsilon_2 s_2' :: G_2'')[v_2/x]$$

or what is the same $\Omega \vdash \Xi_1 \triangleright s_1'[v_1/x] : G_1''' \leqslant \Xi_2 \triangleright s_2'[v_2/x] : G_2'''$. But the result follows immediately by the induction hypothesis on $\Omega \vdash \Xi_1 \triangleright s_1' : G_1''' \leqslant \Xi_2 \triangleright s_2' : G_2'''$.

Case $(\varepsilon_{G_1'} t_1' :: G_1 \leqslant \varepsilon_{G_2'} t_2' :: G_2')$. We know that

$$\frac{\varepsilon_{G_1} \nleqslant \varepsilon_{G_2} \qquad \Omega \vdash \Xi_1 \triangleright t_1' : G_1''' \leqslant \Xi_2 \triangleright t_2' : G_2''' \qquad G_1' \sqsubseteq G_2' \qquad G_1'' \rightarrow G_1' \qquad G_2'' \rightarrow G_2'}{\Omega \vdash \Xi_1 \triangleright \varepsilon_{G_1'} t_1' :: G_1' : G_1' \leqslant \Xi_2 \triangleright \varepsilon_{G_2'} t_2' :: G_2' : G_2'}$$

Note that we are required to prove that

$$(\varepsilon_{G_1'} t_1' :: G_1')[v_1/x] = (\varepsilon_{G_1''} t_1'[v_1/x] :: G_1') \leqslant$$
$$(\varepsilon_{G_2'} t_2'[v_2/x] :: G_2'') = (\varepsilon_{G_2'} t_2' :: G_2')[v_2/x]$$

or what is the same $\Omega \vdash \Xi_1 \triangleright t_1'[v_1/x] : G_1''' \leqslant \Xi_2 \triangleright t_2'[v_2/x] : G_2'''$. But the result follows immediately by the induction hypothesis on $\Omega \vdash \Xi_1 \triangleright t_1' : G_1''' \leqslant \Xi_2 \triangleright t_2' : G_2'''$.

□

Proposition 5.14. *If* $G_1^* \leqslant G_2^*$ *then* $G_1^* \sqsubseteq G_2^*$.

Proof. Examining $\leqslant$ rules.

Case $(B \leqslant B)$. The results follows immediately by the rule $G \sqsubseteq G$.

Case $(X \leqslant X)$. The results follows immediately by the rule $G \sqsubseteq G$.

Case $(\alpha \leqslant \alpha)$. The results follows immediately by the rule $G \sqsubseteq G$.

Case $(B \leqslant ?)$. The results follows immediately by the rule $G \sqsubseteq ?$.

Case $(G_1 \rightarrow G_2 \leqslant ?)$. The results follows immediately by the rule $G \sqsubseteq ?$.

Case $(G_1 \times G_2 \leqslant ?)$. The results follows immediately by the rule $G \sqsubseteq ?$.

Case $(? \leqslant ?)$. The results follows immediately by the rule $G \sqsubseteq ?$.

Case $(\forall X.G_1 \leqslant \forall X.G_2)$. We know that

$$\frac{G_1 \leqslant G_2}{\forall X.G_1 \leqslant \forall X.G_2}$$

By the induction hypothesis on $G_1 \leqslant G_2$, we know that $G_1 \sqsubseteq G_2$. We are required to prove that $\forall X.G_1 \sqsubseteq \forall X.G_2$, which follows immediately by the rule

$$\frac{G_1 \sqsubseteq G_2}{\forall X.G_1 \sqsubseteq \forall X.G_2}$$

Case ($G_1 \rightarrow G_2 \leqslant G_3 \rightarrow G_4$). We know that

$$\frac{G_1 \leqslant G_3 \qquad G_2 \leqslant G_4}{G_1 \rightarrow G_2 \leqslant G_3 \rightarrow G_4}$$

By the induction hypothesis on $G_1 \leqslant G_3$ and $G_2 \leqslant G_4$, we know that $G_1 \sqsubseteq G_3$ and $G_2 \sqsubseteq G_4$. We are required to prove that $G_1 \rightarrow G_2 \sqsubseteq G_3 \rightarrow G_4$, which follows immediately by the rule

$$\frac{G_1 \sqsubseteq G_3 \qquad G_2 \sqsubseteq G_4}{G_1 \rightarrow G_2 \sqsubseteq G_3 \rightarrow G_4}$$

Case ($G_1 \times G_2 \leqslant G_3 \times G_4$). We know that

$$\frac{G_1 \leqslant G_3 \qquad G_2 \leqslant G_4}{G_1 \times G_2 \leqslant G_3 \times G_4}$$

By the induction hypothesis on $G_1 \leqslant G_3$ and $G_2 \leqslant G_4$, we know that $G_1 \sqsubseteq G_3$ and $G_2 \sqsubseteq G_4$. We are required to prove that $G_1 \times G_2 \sqsubseteq G_3 \times G_4$, which follows immediately by the rule

$$\frac{G_1 \sqsubseteq G_3 \qquad G_2 \sqsubseteq G_4}{G_1 \times G_2 \sqsubseteq G_3 \times G_4}$$

□

Proposition 5.15. *If $v_1 \leqslant t_2$ then $t_2 = v_2$.*

Proof. Exploring $\leqslant$ rules. □

Proposition 5.16. *If $\varepsilon_1 \leqslant \varepsilon_2$ then*

- $dom(\varepsilon_1) \leqslant dom(\varepsilon_2)$
- $cod(\varepsilon_1) \leqslant cod(\varepsilon_2)$
- $p_i(\varepsilon_1) \leqslant p_i(\varepsilon_2)$
- $schm_u(\varepsilon_1) \leqslant schm_u(\varepsilon_2)$

Proof. By inspecting the evidence shape and the definition of $\varepsilon_1 \leqslant \varepsilon_2$. □

Proposition 5.17. *If $\varepsilon \Vdash \Xi; \Delta \vdash G'' \sim G'$ and $G' \rightharpoonup G$, then $\varepsilon \mathbin{;} \varepsilon_G = \varepsilon$.*

Proof. By Lemma 6.30 and definition of $G' \rightharpoonup G$ and $\varepsilon \mathbin{;} \varepsilon_G = \varepsilon$. □

Proposition 5.18. *If $\Xi_1 \vdash t_1 \leqslant \Xi_2 \vdash t_2$ and $\Xi_1 \triangleright t_1 \longrightarrow \Xi_1' \triangleright t_1'$, then $\Xi_2 \triangleright t_2 \longrightarrow \Xi_2' \triangleright t_2'$ and $\Xi_1' \vdash t_1' \leqslant \Xi_2' \vdash t_2'$.*

Proof. If $\Xi_1 \vdash t_1 \leqslant \Xi_2 \vdash t_2$, we know that $\vdash t_1 \leqslant t_2 : G_1 \leqslant G_2$, $\Xi_1 \leqslant \Xi_2$, $\Xi_1 \vdash t_1 : G_1$ and $\Xi_2 \vdash t_2 : G_2$. We follow by induction on $\vdash t_1 \leqslant t_2 : G_1 \leqslant G_2$. We avoid the notation $\vdash t_1 \leqslant t_2 : G_1 \leqslant G_2$, and use $t_1 \leqslant t_2$ instead, for simplicity, when the typing environments are not relevant.

Case ($b \leqslant b$). This case does not applies because b is not a term t, therefore it can not reduce.

Case ($x \leqslant x$). This case does not applies because x is not a term t, therefore it can not reduce.

Case ($(\lambda x : G_1^*.t_1^*) \leqslant (\lambda x : G_2^*.t_2^*)$). This case does not applies because $\lambda x : G_1^*.t_1^*$ is not a term t, therefore it can not reduce.

Case ($(\Lambda X.t_1^*) \leqslant (\Lambda X.t_2^*)$). This case does not applies because $\Lambda X.t_1^*$ is not a term t, therefore it can not reduce.

Case $(t^*_{11}\ t^*_{12} \leqslant t^*_{21}\ t^*_{22})$. We know that

$$\frac{t^*_{11} \leqslant t^*_{21} \qquad t^*_{12} \leqslant t^*_{22}}{t^*_{11}\ t^*_{12} \leqslant t^*_{21}\ t^*_{22}}$$

Also, since $\Xi_1 \triangleright t_1 \longrightarrow \Xi'_1 \triangleright t'_1$, we know that $t^*_{11} = \varepsilon_{11}\lambda x : G^*_1.t_{11} :: G_{12} \to G_{11}$ and $t^*_{12} = v_{12} = \varepsilon_{12}u_{12} : G_{12}$. By Proposition 5.15, we know that $t^*_{21} = \varepsilon_{21}\lambda x : G^*_2.t_{21} :: G_{22} \to G_{21}$ and $t^*_{22} = v_{22} = \varepsilon_{22}u_{22} : G_{22}$. By the reduction rules, we know that

$$\Xi_1 \triangleright (\varepsilon_{11}\lambda x : G^*_1.t_{11} :: G_{12} \to G_{11})\ (\varepsilon_{12}u_{12} : G_{12}) \longrightarrow \Xi_1 \triangleright cod(\varepsilon_{11})(t_{11}[((\varepsilon_{12} \circ dom(\varepsilon_{11}))u_{11} :: G^*_1)/x]) :: G_{11}$$

By Proposition 5.16, we know that $dom(\varepsilon_{11}) \leqslant dom(\varepsilon_{21})$ and $cod(\varepsilon_{11}) \leqslant cod(\varepsilon_{21})$. Therefore, by Proposition **??** and $\varepsilon_{12} \leqslant \varepsilon_{22}$, we know that $(\varepsilon_{12} \circ dom(\varepsilon_{11})) \leqslant (\varepsilon_{22} \circ dom(\varepsilon_{21}))$.

Therefore, we know that

$$\Xi_2 \triangleright (\varepsilon_{21}\lambda x : G^*_2.t_{21} :: G_{22} \to G_{21})\ (\varepsilon_{22}u_{22} : G_{22}) \longrightarrow \Xi_2 \triangleright cod(\varepsilon_{21})(t_{21}[((\varepsilon_{22} \circ dom(\varepsilon_{21}))u_{21} :: G^*_2)/x]) :: G_{21}$$

Thus, by the $\leqslant$ rules, $u_{11} \leqslant u_{21}$ and $G^*_1 \sqsubseteq G^*_2$, we know that

$$((\varepsilon_{12} \circ dom(\varepsilon_{11}))u_{11} :: G^*_1) \leqslant ((\varepsilon_{22} \circ dom(\varepsilon_{21}))u_{21} :: G^*_2)$$

By Proposition 5.13, we know that

$$(t_{11}[((\varepsilon_{12} \circ dom(\varepsilon_{11}))u_{11} :: G^*_1)/x]) \leqslant (t_{21}[((\varepsilon_{22} \circ dom(\varepsilon_{21}))u_{21} :: G^*_2)/x])$$

Finally, since $cod(\varepsilon_{11}) \leqslant cod(\varepsilon_{21})$ and $G_{11} \sqsubseteq G_{21}$ and the $\leqslant$ rules the result holds.

$$\Xi_1 \vdash cod(\varepsilon_{11})(t_{11}[((\varepsilon_{12} \circ dom(\varepsilon_{11}))u_{11} :: G^*_1)/x]) :: G_{11} \leqslant \Xi_2 \vdash cod(\varepsilon_{21})(t_{21}[((\varepsilon_{22} \circ dom(\varepsilon_{21}))u_{21} :: G^*_2)/x]) :: G_{21}$$

Case $(t^*_1\ [G^*_1] \leqslant t^*_2\ [G^*_2])$. We know that

$$\frac{t^*_1 \leqslant t^*_2 \qquad G^*_1 \leqslant G^*_2}{t^*_1\ [G^*_1] \leqslant t^*_2\ [G^*_2]}$$

Also, since $\Xi_1 \triangleright t_1 \longrightarrow \Xi'_1 \triangleright t'_1$, we know that $t^*_1 = \varepsilon_{11}\Lambda X.t_{11} :: \forall X.G_{11}$. By Proposition 5.15, we know that $t^*_2 = \varepsilon_{22}\Lambda X.t_{22} :: \forall X.G_{22}$. By the reduction rules, we know that

$$\Xi_1 \triangleright (\varepsilon_{11}\Lambda X.t_{11} :: \forall X.G_{11})[G^*_1] \longrightarrow \Xi'_1 \triangleright \varepsilon_{11out}(\varepsilon_{11}[\hat{\alpha}_1]t_{11}[\hat{\alpha}_1/X] :: G_{11}[\alpha/X]) :: G_{11}[G^*_1/X]$$

where $\Xi'_1 = \Xi_1, \alpha := G^*_1$ and $\hat{\alpha}_1 = lift_{\Xi'_1}(\alpha)$.

By Proposition 5.7, we know that $\varepsilon_{11out} \leqslant \varepsilon_{22out}$ and $\varepsilon_{11}[\hat{\alpha}_1] \leqslant \varepsilon_{22}[\hat{\alpha}_2]$.

Therefore, we know that

$$\Xi_2 \triangleright (\varepsilon_{22}\Lambda X.t_{22} :: \forall X.G_{22})[G^*_2] \longrightarrow \Xi'_2 \triangleright \varepsilon_{22out}(\varepsilon_{22}[\hat{\alpha}_2]t_{22}[\hat{\alpha}_2/X] :: G_{22}[\alpha/X]) :: G_{22}[G^*_2/X]$$

where $\Xi'_2 = \Xi_2, \alpha := G^*_2$ and $\hat{\alpha}_2 = lift_{\Xi'_2}(\alpha)$.

By Proposition 5.12 we know that $t_{11}[\hat{\alpha}_1/X] \leqslant t_{22}[\hat{\alpha}_2/X]$. By Proposition 5.8 and Proposition 5.9, we know that $G_{11}[\alpha/X] \leqslant G_{22}[\alpha/X]$ and $G_{11}[G^*_1/X] \leqslant G_{22}[G^*_2/X]$, respectively.

Finally, by the $\leqslant$ rules the result holds.

$$\Xi'_1 \triangleright \varepsilon_{1out}(\varepsilon_1[\hat{\alpha}_1]t_1[\hat{\alpha}_1/X] :: G_1[\alpha/X]) :: G_1[G^*_1/X] \leqslant \Xi'_2 \triangleright \varepsilon_{22out}(\varepsilon_{22}[\hat{\alpha}_2]t_{22}[\hat{\alpha}_2/X] :: G_{22}[\alpha/X]) :: G_{22}[G^*_2/X]$$

Case $(\varepsilon_1 s_1 :: G_1^* \leqslant \varepsilon_2 s_2 :: G_2^*)$. We know that

$$\frac{\varepsilon_1 \leqslant \varepsilon_2 \qquad s_1 \leqslant s_2 \qquad G_1^* \sqsubseteq G_2^*}{\varepsilon_1 s_1 :: G_1^* \leqslant \varepsilon_2 s_2 :: G_2^*}$$

Also, since $\Xi_1 \triangleright t_1 \longrightarrow \Xi_1' \triangleright t_1'$, we know that $s_1 = (\varepsilon_{11} u_{11} :: G_{11})$. By Proposition 5.15, we know that $s_2 = (\varepsilon_2 u_2 :: G_2)$. By the reduction rules, we know that

$$\Xi_1 \triangleright \varepsilon_1(\varepsilon_{11} u_{11} :: G_{11}) :: G_1^* \longrightarrow \Xi_1 \triangleright (\varepsilon_{11} \mathbin{;} \varepsilon_1) u_{11} :: G_1^*$$

By the $\leqslant$ rules, we know that $\varepsilon_{11} \leqslant \varepsilon_{22}$ and $\varepsilon_1 \leqslant \varepsilon_2$. Therefore, by Proposition **??**, we know that $(\varepsilon_{11} \mathbin{;} \varepsilon_1) \leqslant (\varepsilon_{22} \mathbin{;} \varepsilon_2)$.

Therefore, we know that

$$\Xi_2 \triangleright \varepsilon_2(\varepsilon_{22} u_{22} :: G_{22}) :: G_2^* \longrightarrow \Xi_2 \triangleright (\varepsilon_{22} \mathbin{;} \varepsilon_2) u_{22} :: G_2^*$$

Thus, by the $\leqslant$ rules, $u_{11} \leqslant u_{22}$ and $G_1^* \sqsubseteq G_2^*$, the result holds.

$$\Xi_1 \vdash (\varepsilon_{11} \mathbin{;} \varepsilon_1) u_{11} :: G_1^* \leqslant \Xi_2 \vdash (\varepsilon_{22} \mathbin{;} \varepsilon_2) u_{22} :: G_2^*$$

Case $(\varepsilon_{G_1^*}(\varepsilon_{11} u_1 :: G_1^*) :: G_1^* \leqslant \varepsilon_{G_2^*}(\varepsilon_{22} u_2 :: G_2^*) :: G_2^*)$. We know that

$$\frac{\Omega \vdash u_{11} \leqslant u_{22} : G_1^{**} \leqslant G_2^{**} \qquad G_1^* \sqsubseteq G_2^* \qquad G_1^{**} \rightharpoonup G_1^* \qquad G_2^{**} \rightharpoonup G_2^*}{\Omega \vdash \varepsilon_{G_1^*}(\varepsilon_{11} u_{11} :: G_1^{**}) :: G_1^* \leqslant \varepsilon_{G_2^*}(\varepsilon_{22} u_{22} :: G_2^{**}) :: G_2^* : G_1^* \leqslant G_2^*}$$

Also, since $\Xi_1 \triangleright t_1 \longrightarrow \Xi_1' \triangleright t_1'$, we know that $t_1 = \varepsilon_{G_1^*}(\varepsilon_{11} u_{11} :: G_1^{**}) :: G_1^*$. By Proposition 5.15, we know that $s_2 = \varepsilon_{G_2^*}(\varepsilon_{22} u_{22} :: G_2^{**}) :: G_2^*$. By the reduction rules, we know that

$$\Xi_1 \triangleright \varepsilon_{G_1^*}(\varepsilon_{11} u_{11} :: G_1^{**}) :: G_1^* \longrightarrow \Xi_1 \triangleright (\varepsilon_{11} \mathbin{;} \varepsilon_{G_1^*}) u_{11} :: G_1^*$$

We know by the definition of $\Xi \vdash \varepsilon_{G_1^*}(\varepsilon_{11} u_{11} :: G_1^{**}) :: G_1^* \leqslant \varepsilon_{G_2^*}(\varepsilon_{22} u_{22} :: G_2^{**}) :: G_2^* : G_1^* \leqslant G_2^*$ that $\Xi_1 \vdash (\varepsilon_{11} u_{11} :: G_1^{**}) : G_1^{**}$ and $\Xi_2 \vdash (\varepsilon_{22} u_{22} :: G_2^{**}) : G_2^{**}$, and therefore, $\varepsilon_{11} \Vdash \Xi_1 \vdash G_1^{***} \sim G_1^{**}$ and $\varepsilon_{22} \Vdash \Xi_2 \vdash G_2^{***} \sim G_2^{**}$. By the $\leqslant$ rules, we know that $\varepsilon_{11} \leqslant \varepsilon_{22}$ and $\varepsilon_{G_1^*} \sqsubseteq \varepsilon_{G_2^*}$. Therefore, by Lemma 5.17 and $G_1^{**} \rightharpoonup G_1^*$ and $G_2^{**} \rightharpoonup G_2^*$, we know that $(\varepsilon_{11} \mathbin{;} \varepsilon_{G_1^*}) = \varepsilon_{11}$ and $(\varepsilon_{22} \mathbin{;} \varepsilon_{G_2^*}) = \varepsilon_{22}$.

Therefore, we know that

$$\Xi_2 \triangleright \varepsilon_{G_2^*}(\varepsilon_{22} u_{22} :: G_2^{**}) :: G_2^* \longrightarrow \Xi_2 \triangleright (\varepsilon_{22} \mathbin{;} \varepsilon_{G_2^*}) u_{22} :: G_2^*$$

Then, by the $\leqslant$ rules, $u_{11} \leqslant u_{22}$ and $G_1^* \sqsubseteq G_2^*$, the result holds.

$$\Xi_1 \vdash (\varepsilon_{11} \mathbin{;} \varepsilon_1) u_{11} :: G_1^* \leqslant \Xi_2 \vdash (\varepsilon_{22} \mathbin{;} \varepsilon_2) u_{22} :: G_2^*$$

□

Proposition 5.19. *If* $\Xi_1 \vdash t_1 \leqslant \Xi_2 \vdash t_2$ *and* $\Xi_1 \triangleright t_1 \longmapsto \Xi_1' \triangleright t_1'$, *then* $\Xi_2 \triangleright t_2 \longmapsto \Xi_2' \triangleright t_2'$ *and* $\Xi_1' \vdash t_1' \leqslant \Xi_2' \vdash t_2'$.

Proof. If $\Xi_1 \vdash t_1 \leqslant \Xi_2 \vdash t_2$, we know that $\vdash t_1 \leqslant t_2 : G_1 \leqslant G_2$, $\Xi_1 \leqslant \Xi_2$, $\Xi_1 \vdash t_1 : G_1$ and $\Xi_2 \vdash t_2 : G_2$. We avoid the notation $\vdash t_1 \leqslant t_2 : G_1 \leqslant G_2$, and use $t_1 \leqslant t_2$ instead, for simplicity, when the typing environments are not relevant.

By induction on reduction $\Xi_1 \triangleright t_1 \longmapsto \Xi_1' \triangleright t_1'$.

Case $(\Xi_1 \triangleright t_1 \longrightarrow \Xi_1' \triangleright t_1')$. By Proposition 5.18, we know that $\Xi_2 \triangleright t_2 \longrightarrow \Xi_2' \triangleright t_2'$, $\Xi_1' \vdash t_1' \leqslant \Xi_2' \vdash t_2'$; and the result holds immediately.

Case ($\Xi_1 \triangleright \varepsilon_{11} t_{11} :: G_{11} \longmapsto \Xi_1' \triangleright \varepsilon_{11} t_{11}' :: G_{11}$). By inspection of $\leqslant$, $t_2 = \varepsilon_{22} t_{22} :: G_{22}$, where $\varepsilon_{11} \leqslant \varepsilon_{22}$ or $\varepsilon_{11} \sqsubseteq \varepsilon_{22}$, $t_{11} \leqslant t_{22}$ and $G_{11} \sqsubseteq G_{22}$. By induction hypothesis on $\Xi_1 \triangleright t_{11} \longmapsto \Xi_1' \triangleright t_{11}'$, then $\Xi_2 \triangleright t_{22} \longmapsto \Xi_2' \triangleright t_{22}'$, where $\Xi_1' \vdash t_{11}' \leqslant \Xi_2' \vdash t_{22}'$. Then, by $\leqslant$, we know that $\Xi_1' \vdash \varepsilon_{11} t_{11}' :: G_{11} \leqslant \Xi_2' \vdash \varepsilon_{22} t_{22}' :: G_{22}$ and the result holds.

Case ($\Xi_1 \triangleright t_{11}\ t_{12} \longmapsto \Xi_1' \triangleright t_{11}'\ t_{12}$). By inspection of $\leqslant$, $t_2 = t_{21}\ t_{22}$, where $t_{11} \leqslant t_{21}$ and $t_{12} \leqslant t_{22}$. By induction hypothesis on $\Xi_1 \triangleright t_{11} \longmapsto \Xi_1' \triangleright t_{11}'$, we know that $\Xi_2 \triangleright t_{21} \longmapsto \Xi_2' \triangleright t_{21}'$, where $\Xi_1' \vdash t_{11}' \leqslant \Xi_2' \vdash t_{21}'$. Then, by $\leqslant$, we know that $\Xi_1' \vdash t_{11}'\ t_{12} \leqslant \Xi_2' \vdash t_{21}'\ t_{22}$ and the result holds.

Case ($\Xi_1 \triangleright v_{11}\ t_{12} \longmapsto \Xi_1' \triangleright v_{11}\ t_{12}'$). By inspection of $\leqslant$ and Proposition 5.15, $t_2 = v_{21}\ t_{22}$, where $v_{11} \leqslant v_{21}$ and $t_{12} \leqslant t_{22}$. By induction hypothesis on $\Xi_1 \triangleright t_{12} \longmapsto \Xi_1' \triangleright t_{12}'$, then $\Xi_2 \triangleright t_{22} \longmapsto \Xi_2' \triangleright t_{22}'$, where $\Xi_1' \vdash t_{12}' \leqslant \Xi_2' \vdash t_{22}'$. Then, by $\leqslant$, we know that $\Xi_1' \vdash v_{11}\ t_{12}' \leqslant \Xi_2' \vdash v_{21}\ t_{22}'$ and the result holds.

Case ($\Xi_1 \triangleright t_{11}\ [G_{11}] \longmapsto \Xi_1' \triangleright t_{11}'\ [G_{11}]$). By inspection of $\leqslant$, $t_2 = t_{22}\ [G_{22}]$, where $t_{11} \leqslant t_{22}$ and $G_{11} \leqslant G_{22}$. By induction hypothesis on $\Xi_1 \triangleright t_{11} \longmapsto \Xi_1' \triangleright t_{11}'$, we know that $\Xi_2 \triangleright t_{22} \longmapsto \Xi_2' \triangleright t_{22}'$, where $\Xi_1' \vdash t_{11}' \leqslant \Xi_2' \vdash t_{22}'$. Then, by $\leqslant$, we know that $\Xi_1' \vdash t_{11}'\ [G_{11}] \leqslant \Xi_2' \vdash t_{22}'\ [G_{22}]$ and the result holds.

Proposition 9.4 (Small-step DGG$^{\leqslant}$ for GSFε). *Suppose* $\Xi_1 \triangleright t_1 \leqslant \Xi_2 \triangleright t_2$.

a. If $\Xi_1 \triangleright t_1 \longmapsto \Xi_1' \triangleright t_1'$, *then* $\Xi_2 \triangleright t_2 \longmapsto \Xi_2' \triangleright t_2'$, *and we have* $\Xi_1' \triangleright t_1' \leqslant \Xi_2' \triangleright t_2'$.
b. If $t_1 = v_1$, *then* $t_2 = v_2$.

Proof. Direct by Lemma 5.19 and 5.15. □

□

Proposition 5.20. *Let suppose* $\Xi_1 \vdash t_1 \leqslant \Xi_2 \vdash t_2$.

- $\Xi_1 \triangleright t_1 \longmapsto^* \Xi_1' \triangleright v_1$ *implies* $\Xi_2 \triangleright t_2 \longmapsto^* \Xi_2' \triangleright v_2, \Xi_1' \vdash v_1 \leqslant \Xi_2' \vdash v_2$.
- t_1 *diverges implies* t_2 *diverges.*
- $\Xi_2 \triangleright t_2 \longmapsto^* \Xi_2' \triangleright v_2$ *implies* $\Xi_1 \triangleright t_1 \longmapsto^* \Xi_1' \triangleright v_1$ *and* $\Xi_1' \vdash v_1 \leqslant \Xi_2' \vdash v_2$, *or* $\Xi_1 \triangleright t_1 \longmapsto^*$ **error**.
- t_2 *diverges implies* t_1 *diverges, or* $\Xi_1 \triangleright t_1 \longmapsto^*$ **error**.

Proof. The proof is by case analysis on the reduction of t_1 or t_2.

- Suppose that $\Xi_1 \triangleright t_1 \longmapsto^* \Xi_1' \triangleright v_1$. Then $\Xi_2 \triangleright t_2 \longmapsto^* \Xi_2' \triangleright v_2$, $\Xi_1' \vdash v_1 \leqslant \Xi_2' \vdash v_2$ by Proposition 5.19 and Proposition 5.15.
- Suppose that t_1 diverges. Then t_2 diverges by Proposition 5.19.
- Suppose that $\Xi_2 \triangleright t_2 \longmapsto^* \Xi_2' \triangleright v_2$. Then, the only possibilities given the two previous results are $\Xi_1 \triangleright t_1 \longmapsto^* \Xi_1' \triangleright v_1$ and $\Xi_1' \vdash v_1 \leqslant \Xi_2' \vdash v_2$, or $\Xi_1 \triangleright t_1 \longmapsto^*$ **error**, and the result holds.
- Suppose that t_2 diverges. Then, the only possibilities given the two previous results are t_1 diverges, or $\Xi_1 \triangleright t_1 \longmapsto^*$ **error**, and the result holds.

□

Theorem 9.5 (DGG$^{\leqslant}$). *Suppose* $t_1 \leqslant t_2$, $\vdash t_1 : G_1$, *and* $\vdash t_2 : G_2$.

a. If $t_1 \Downarrow v_1$, *then* $t_2 \Downarrow v_2$ *and* $\cdot \vdash \Xi_1 \triangleright v_1 : G_1 \leqslant \Xi_2 \triangleright v_2 : G_2$, *for some* $\Xi_1 \leqslant \Xi_2$.
If $t_1 \Uparrow$ *then* $t_2 \Uparrow$.
b. If $t_2 \Downarrow v_2$, *then* $t_1 \Downarrow v_1$ *and* $\cdot \vdash \Xi_1 \triangleright v_1 : G_1 \leqslant \Xi_2 \triangleright v_2 : G_2$, *for some* $\Xi_1 \leqslant \Xi_2$, *or* $t_1 \Downarrow$ **error**.
If $t_2 \Uparrow$, *then* $t_1 \Uparrow$ *or* $t_1 \Downarrow$ **error**.

Proof. Direct by Lemma 5.26 and 5.20. □

Lemma 9.6. *Let* $\vdash t : G$, $G \sqsubseteq G'$, *and* $t' = t :: G' :: G$, *then*

- $t \Downarrow v \iff t' \Downarrow v$
- $t \Downarrow \textbf{error} \iff t' \Downarrow \textbf{error}$

Proof. Direct consequence of the weak dynamic gradual guarantee (Theorem 9.5). □

Lemma 9.8. *Let* $\vdash t_1 : G_1$ *and* $\vdash t_2 : G_2$ *such that* $\vdash t_1\ t_2 : G$, $t_1\ t_2 \Downarrow v$, *and let* $G_1 \sqsubseteq G_1'$, $G_2 \sqsubseteq G_2'$, *and* $G \sqsubseteq G'$, *such that* $\vdash (t_1 :: G_1')\ (t_2 :: G_2') : G'$, *then* $(t_1 :: G_1')\ (t_2 :: G_2') \Downarrow v'$ *such that* $\vdash \Xi_1 \triangleright v : G \leqslant \Xi_2 \triangleright v' : G'$, *for some* Ξ_1, Ξ_2.

Proof. From $\vdash (t_1 :: G_1')\ (t_2 :: G_2') : G$, we know that $\vdash G_1 \sim G_1'$ and $\vdash G_2 \sim G_2'$, where $\vdash t_1 : G_1$ ad $\vdash t_2 : G_2$. As $G_1 \sqsubseteq G_1'$, and $G_2 \sqsubseteq G_2'$, then $G_1 \sqcap G_1 \sqsubseteq G_1 \sqcap G_1'$ and $G_2 \sqcap G_2 \sqsubseteq G_2 \sqcap G_2'$. Notice that if $t_1\ t_2 \Downarrow v$, then $(t_1 :: G_1)\ (t_2 :: G_2) \Downarrow v$ (trivial ascriptions). Therefore, by ($\leqslant$ascv) or ($\leqslant$asct), $\vdash (t_1 :: G_1)\ (t_2 :: G_2) : T \leqslant (t_1 :: G_1')\ (t_2 :: G_2') : G$, then the result holds by DGG$^{\leqslant}$ (Th.9.5). □

Lemma 9.7. *Let* $\vdash t : G$ *such that* $t \Downarrow v$, *and let* $G \sqsubseteq G'$, *then* $t :: G' \Downarrow v'$ *such that* $\vdash \Xi \triangleright v : G \leqslant \Xi \triangleright v' : G'$, *for some* Ξ.

Direct by Th.9.5. Similar to Lemma 9.8.

Lemma 9.9. *Let* $\vdash t : G_1$ *such that* $\vdash t\ [G_2] : G$, $t\ [G_2] \Downarrow v$, *and let* $G_1 \sqsubseteq G_1'$, $G_2 \leqslant G_2'$, *and* $G \sqsubseteq G'$, *such that* $\vdash (t :: G_1')\ [G_2'] : G'$, *then* $(t :: G_1')\ [G_2'] \Downarrow v'$ *such that* $\vdash \Xi_1 \triangleright v : G \leqslant \Xi_2 \triangleright v' : G'$, *for some* Ξ_1, Ξ_2.

Proof. Direct by Th.9.5. Similar to Lemma 9.8. □

Proposition 9.10. *Suppose* t_1 *and* t_2 *GSF terms such that* $\cdot \vdash t_1 : G_1 \leqslant t_2 : G_2$, *and their elaborations* $\cdot \vdash t_1 \rightsquigarrow t_{\varepsilon_1} : G_1$ *and* $\cdot \vdash t_2 \rightsquigarrow t_{\varepsilon_2} : G_2$. *Then* $\cdot \vdash \cdot \triangleright t_{\varepsilon_1} : G_1 \leqslant \cdot \triangleright t_{\varepsilon_2} : G_2$.

Proof. Direct by Prop. 5.26. □

5.4 Syntactic Strict Precision for GSF

Now, we present the proof of the weak dynamic gradual guarantee for GSF previously presented and the auxiliary Propositions an Definitions.

Proposition 5.21. $\mathcal{I}_\Xi(G_1 \sqcap G_2, G_1 \sqcap G_2) = \mathcal{I}_\Xi(G_1, G_2)$

Proof. By the definition of $\sqcap$ and $\mathcal{I}_\Xi(G_1, G_2)$. □

Proposition 5.22. $\Omega \vdash s_1 \leqslant s_2 : G_1 \leqslant G_2$ *then* $G_1 \sqsubseteq G_2$.

Proof. By the definition of $\sqcap$ and $\mathcal{I}_\Xi(G_1, G_2)$. □

Proposition 5.23. *If* $G_1 \sqcap G_2 \leqslant G_1' \sqcap G_2'$, *then*

$$\mathcal{I}_\Xi(G_1, G_2) = \mathcal{I}_\Xi(G_1 \sqcap G_2, G_1 \sqcap G_2) \leqslant \mathcal{I}_\Xi(G_1' \sqcap G_2', G_1' \sqcap G_2') = \mathcal{I}_\Xi(G_1', G_2')$$

Proof. By Proposition 5.21 and the definition of $\leqslant$ in evidence. □

Proposition 5.24. *If* $G_1 \leqslant G_2$, *then*

$$\mathcal{I}_\Xi(G_1, G_1) \sqsubseteq \mathcal{I}_\Xi(G_2, G_2)$$

Proof. By the definition of $\mathcal{I}_\Xi$ and the $\sqsubseteq$ in evidence. □

Definition 5.25. $\Omega \equiv \Gamma_1 \sqsubseteq \Gamma_2 \iff (\Omega = \Omega', x : G_1 \sqsubseteq G_2, \Gamma_1 = \Gamma_1', x : G_1, \Gamma_2 = \Gamma_2', x : G_2, G_1 \sqsubseteq G_2$ and $\Omega' \equiv \Gamma_1' \sqsubseteq \Gamma_2') \vee (\Omega = \Gamma_1 = \Gamma_2 = \cdot)$.

Proposition 5.26. *If* $\Omega \vdash \Xi_1 \triangleright t_1^* : G_1^* \leqslant \Xi_2 \triangleright t_2^* : G_2^*$, $\Omega \equiv \Gamma_1 \sqsubseteq \Gamma_2$, $\Xi_1 \leqslant \Xi_2$ *and* $\Xi_i; \Delta; \Gamma_i \vdash t_i^* \rightsquigarrow t_i^{**} : G_i^*$, *then* $\Omega \vdash \Xi_1 \triangleright t_1^{**} : G_1^* \leqslant \Xi_2 \triangleright t_2^{**} : G_2^*$.

Proof. We follow by induction on $\Omega \vdash \Xi_1 \triangleright t_1^* : G_1^* \leqslant \Xi_2 \triangleright t_2^* : G_2^*$. We avoid the notation $\Omega \vdash \Xi_1 \triangleright t_1^* : G_1^* \leqslant \Xi_2 \triangleright t_2^* : G_2^*$, and use $t_1^* \leqslant t_2^*$ instead, for simplicity, when the typing environments are not relevant. We use metavariable v or u in GSF to range over constants, functions and type abstractions.

Remember that

$$norm(t, G_1, G_2) = \varepsilon t :: G_2, \text{ where } \varepsilon = \mathcal{I}_\Xi(G_1, G_2)$$

By Proposition 5.21 we know that

$$\mathcal{I}_\Xi(G_1, G_2) = \mathcal{I}_\Xi(G_1 \sqcap G_2, G_1 \sqcap G_2) = \mathcal{I}(lift_\Xi(G_1), lift_\Xi(G_2))$$

Case $(\Omega \vdash \Xi_1 \triangleright u_1 : G_1^* \leqslant \Xi_2 \triangleright u_2 : G_2^*)$. We know that

$$(\leqslant\text{v})\frac{\Omega \vdash u_1 : G_1^* \leqslant_v u_2 : G_2^* \qquad G_1^* \leqslant G_2^*}{\Omega \vdash \Xi_1 \triangleright u_1 : G_1^* \leqslant \Xi_2 \triangleright u_2 : G_2^*}$$

$$(\text{G}u)\frac{\Xi_1; \Delta; \Gamma_1 \vdash u_1 \rightsquigarrow u_1' : G_1^* \qquad \varepsilon_{G_1^*} = \mathcal{I}_\Xi(G_1^*, G_1^*)}{\Xi_1; \Delta; \Gamma_1 \vdash u_1 \rightsquigarrow \varepsilon_{G_1^*} u_1' :: G_1^* : G_1^*}$$

$$(\text{G}u)\frac{\Xi_2; \Delta; \Gamma_2 \vdash u_2 \rightsquigarrow u_2' : G_2^* \qquad \varepsilon_{G_2^*} = \mathcal{I}_\Xi(G_2^*, G_2^*)}{\Xi_2; \Delta; \Gamma_2 \vdash u_2 \rightsquigarrow \varepsilon_{G_2^*} u_2' :: G_2^* : G_2^*}$$

We have to prove that $\Omega \vdash \varepsilon_{G_1^*} u_1' :: G_1^* \leqslant \varepsilon_{G_2^*} u_2' :: G_2^* : G_1^* \leqslant G_2^*$. By the rule ($\leqslant$asc$_\varepsilon$), we are required to prove that $\varepsilon_{G_1^*} \leqslant \varepsilon_{G_2^*}$, $\Omega \vdash u_1' \leqslant u_2' : G_1^* \leqslant G_2^*$ and $G_1^* \sqsubseteq G_2^*$. Since $G_1^* \leqslant G_2^*$, $\Xi_1 \leqslant \Xi_2$ and Proposition 5.3, we know that $\varepsilon_{G_1^*} \leqslant \varepsilon_{G_2^*}$. Also, by Proposition 5.14 and $G_1^* \leqslant G_2^*$ we now that $G_1^* \sqsubseteq G_2^*$. Therefore, we only have required to prove that $\Omega \vdash u_1' \leqslant u_2' : G_1^* \leqslant G_2^*$. We follow by case analysis on $\Omega \vdash u_1 : G_1^* \leqslant_v u_2 : G_2^*$.

- Case $(\Omega \vdash b : B \leqslant_v b : B)$. We know that

$$(\leqslant b)\frac{ty(b) = B}{\Omega \vdash b : B \leqslant_v b : B}$$

$$(\text{G}b)\frac{ty(b) = B}{\Xi_i; \Delta; \Gamma_i \vdash b \rightsquigarrow b : B}$$

  We have to prove that $\Omega \vdash b \leqslant b : B \leqslant B$. Then, by ($\leqslant b_\varepsilon$) rule, we know that $\Omega \vdash b \leqslant b : B \leqslant B$ and the result holds.
- Case $(\Omega \vdash (\lambda x : G_1.t_1) : G_1 \to G_2 \leqslant_v (\lambda x : G_1'.t_2) : G_1' \to G_2')$. We know that

$$(\leqslant\lambda)\frac{\Omega, x : G_1 \sqsubseteq G_1' \vdash \Xi_1 \triangleright t_1 : G_2 \leqslant \Xi_2 \triangleright t_2 : G_2' \qquad G_1 \sqsubseteq G_1'}{\Omega \vdash (\lambda x : G_1.t_1) : G_1 \to G_2 \leqslant_v (\lambda x : G_1'.t_2) : G_1' \to G_2'}$$

$$(\text{G}\lambda)\frac{\Xi_1; \Delta; \Gamma_1, x : G_1 \vdash t_1 \rightsquigarrow t_1' : G_2}{\Xi_1; \Delta; \Gamma_1 \vdash (\lambda x : G_1.t_1) \rightsquigarrow (\lambda x : G_1.t_1') : G_1 \to G_2}$$

$$(\text{G}\lambda)\frac{\Xi_2; \Delta; \Gamma_2, x : G_1' \vdash t_2 \rightsquigarrow t_2' : G_2'}{\Xi_2; \Delta; \Gamma_2 \vdash (\lambda x : G_1'.t_2) \rightsquigarrow (\lambda x : G_1'.t_2') : G_1' \to G_2'}$$

  Therefore, we are required to prove that $\Omega \vdash (\lambda x : G_1.t_1') \leqslant (\lambda x : G_1'.t_2') : G_1 \to G_2 \leqslant G_1' \to G_2'$, or what is the same by the ($\leqslant\lambda_\varepsilon$) that $\Omega, x : G_1 \sqsubseteq G_1' \vdash t_1' \leqslant t_2' : G_2 \leqslant G_2'$, but the result follows immediately by the induction hypothesis on $\Omega, x : G_1 \sqsubseteq G_1' \vdash \Xi_1 \triangleright t_1 : G_2 \leqslant \Xi_2 \triangleright t_2 : G_2'$, with the translations t_1' and $t_2'(\Omega, x : G_1 \sqsubseteq G_1' \equiv \Gamma_1, x : G_1 \sqsubseteq \Gamma_2, x : G_1')$.

- Case $(\Omega \vdash (\Lambda X.t_1) : \forall X.G_1 \leqslant_v (\Lambda X.t_2) : \forall X.G_2)$. We know that

$$(\leqslant\Lambda)\dfrac{\Omega \vdash \Xi_1 \triangleright t_1 : G_1 \leqslant \Xi_2 \triangleright t_2 : G_2}{\Omega \vdash (\Lambda X.t_1) : \forall X.G_1 \leqslant_v (\Lambda X.t_2) : \forall X.G_2}$$

$$(\mathrm{G}\lambda)\dfrac{\Xi_1; \Delta, X; \Gamma_1 \vdash t_1 \rightsquigarrow t_1' : G_1}{\Xi_1; \Delta; \Gamma_1 \vdash (\Lambda X.t_1) \rightsquigarrow (\Lambda X.t_1') : \forall X.G_1}$$

$$(\mathrm{G}\lambda)\dfrac{\Xi_2; \Delta, X; \Gamma_2 \vdash t_2 \rightsquigarrow t_2' : G_2}{\Xi_2; \Delta; \Gamma_2 \vdash (\Lambda X.t_2) \rightsquigarrow (\Lambda X.t_2') : \forall X.G_2}$$

Therefore, we are required to prove that $\Omega \vdash (\Lambda X.t_1') \leqslant (\Lambda X.t_2') : \forall X.G_1 \leqslant \forall X.G_2$, or what is the same by the rule $(\leqslant\Lambda_\varepsilon)$ that $\Omega \vdash t_1' \leqslant t_2' : G_1 \leqslant G_2$, but the result follows immediately by the induction hypothesis on $\Omega \vdash \Xi_1 \triangleright t_1 : G_1 \leqslant \Xi_2 \triangleright t_2 : G_2$, with the translations t_1' and t_2'.

Case $(\Omega \vdash \Xi_1 \triangleright x : G_1^* \leqslant \Xi_2 \triangleright x : G_2^*)$. We know that

$$(\leqslant\mathrm{x})\dfrac{x : G_1^* \sqsubseteq G_2^* \in \Omega}{\Omega \vdash \Xi_1 \triangleright x : G_1^* \leqslant \Xi_2 \triangleright x : G_2^*}$$

$$(\mathrm{Gx})\dfrac{x : G_1^* \in \Gamma_1}{\Xi_1; \Delta; \Gamma_1 \vdash x \rightsquigarrow x : G_1^*}$$

$$(\mathrm{Gx})\dfrac{x : G_2^* \in \Gamma_2}{\Xi_2; \Delta; \Gamma_2 \vdash x \rightsquigarrow x : G_2^*}$$

We have to prove that $\Omega \vdash x \leqslant x : G_1^* \leqslant G_2^*$. Then, by the rule $(\leqslant\mathrm{x}_\varepsilon)$, we know that $\Omega \vdash x \leqslant x : G_1^* \leqslant G_2^*$ and the result holds.

Case $((\leqslant\mathrm{ascv}))$. We know that

$$(\leqslant\mathrm{ascv})\dfrac{\Omega \vdash u_1 : G_1^{**} \leqslant_v u_2 : G_2^{**} \qquad G_1^{**} \sqcap G_1^* \leqslant G_2^{**} \sqcap G_2^* \qquad G_1^* \sqsubseteq G_2^*}{\Omega \vdash \Xi_1 \triangleright u_1 :: G_1^* : G_1^* \leqslant \Xi_2 \triangleright u_2 :: G_2^* : G_2^*}$$

$$(\mathrm{Gascu})\dfrac{\Xi_1; \Delta; \Gamma_1 \vdash u_1 \rightsquigarrow u_1' : G_1^{**} \qquad \varepsilon_1 = \mathcal{I}_\Xi(G_1^{**}, G_1^*)}{\Xi_1; \Delta; \Gamma_1 \vdash u_1 :: G_1^* \rightsquigarrow \varepsilon_1 u_1' :: G_1^* : G_1^*}$$

$$(\mathrm{Gascu})\dfrac{\Xi_2; \Delta; \Gamma_2 \vdash u_2 \rightsquigarrow u_2' : G_2^{**} \qquad \varepsilon_2 = \mathcal{I}_\Xi(G_2^{**}, G_2^*)}{\Xi_2; \Delta; \Gamma_2 \vdash u_2 :: G_2^* \rightsquigarrow \varepsilon_2 u_2' :: G_2^* : G_2^*}$$

We have to prove that $\Omega \vdash \varepsilon_1 u_1' :: G_1^* \leqslant \varepsilon_2 u_2' :: G_2^* : G_1^* \leqslant G_2^*$, or what is the same by the rule $(\leqslant\mathrm{asc}_\varepsilon)$, we have to prove that $\varepsilon_1 \leqslant \varepsilon_2, \Omega \vdash u_1' \leqslant u_2' : G_1^{**} \leqslant G_2^{**}$ and $G_1^* \sqsubseteq G_2^*$. By Proposition 5.21, we know that $\varepsilon_1 = \mathcal{I}_\Xi(G_1^{**}, G_1^*) = \mathcal{I}_\Xi(G_1^{**} \sqcap G_1^*, G_1^{**} \sqcap G_1^*)$ and $\varepsilon_2 = \mathcal{I}_\Xi(G_2^{**}, G_2^*) = \mathcal{I}_\Xi(G_2^{**} \sqcap G_2^*, G_2^{**} \sqcap G_2^*)$. Since $G_1^{**} \sqcap G_1^* \leqslant G_2^{**} \sqcap G_2^*$, then $\varepsilon_1 = \mathcal{I}_\Xi(G_1^{**}, G_1^*) = \mathcal{I}_\Xi(G_1^{**} \sqcap G_1^*, G_1^{**} \sqcap G_1^*) \leqslant \mathcal{I}_\Xi(G_2^{**} \sqcap G_2^*, G_2^{**} \sqcap G_2^*) = \mathcal{I}_\Xi(G_2^{**}, G_2^*) = \varepsilon_2$, by Proposition 5.23. Thus, we only have to prove that $\Omega \vdash u_1' \leqslant u_2' : G_1^{**} \leqslant G_2^{**}$, and we know that $\Omega \vdash u_1' : G_1^{**} \leqslant_v u_2' : G_2^{**}$. We follow by case analysis on $\Omega \vdash u_1 : G_1^{**} \leqslant_v u_2 : G_2^{**}$.

- Case $(\Omega \vdash b : B \leqslant_v b : B)$. We know that

$$(\leqslant b)\dfrac{ty(b) = B}{\Omega \vdash b : B \leqslant_v b : B}$$

$$(\mathrm{G}b)\dfrac{ty(b) = B}{\Xi_i; \Delta; \Gamma_i \vdash b \rightsquigarrow b : B}$$

We have to prove that $\Omega \vdash b \leqslant b : B \leqslant B$. Then, by $(\leqslant b_\varepsilon)$ rule, we know that $\Omega \vdash b \leqslant b : B \leqslant B$ and the result holds.

- Case $(\Omega \vdash (\lambda x : G_1.t_1) : G_1 \to G_2 \leqslant_v (\lambda x : G'_1.t_2) : G'_1 \to G'_2)$. We know that

$$(\leqslant\lambda)\frac{\Omega, x : G_1 \sqsubseteq G'_1 \vdash \Xi_1 \triangleright t_1 : G_2 \leqslant \Xi_2 \triangleright t_2 : G'_2 \qquad G_1 \sqsubseteq G'_1}{\Omega \vdash (\lambda x : G_1.t_1) : G_1 \to G_2 \leqslant_v (\lambda x : G'_1.t_2) : G'_1 \to G'_2}$$

$$(\mathrm{G}\lambda)\frac{\Xi_1; \Delta; \Gamma_1, x : G_1 \vdash t_1 \rightsquigarrow t'_1 : G_2}{\Xi_1; \Delta; \Gamma_1 \vdash (\lambda x : G_1.t_1) \rightsquigarrow (\lambda x : G_1.t'_1) : G_1 \to G_2}$$

$$(\mathrm{G}\lambda)\frac{\Xi_2; \Delta; \Gamma_2, x : G'_1 \vdash t_2 \rightsquigarrow t'_2 : G'_2}{\Xi_2; \Delta; \Gamma_2 \vdash (\lambda x : G'_1.t_2) \rightsquigarrow (\lambda x : G'_1.t'_2) : G'_1 \to G'_2}$$

Therefore, we are required to prove that $\Omega \vdash (\lambda x : G_1.t'_1) \leqslant (\lambda x : G'_1.t'_2) : G_1 \to G_2 \leqslant G'_1 \to G'_2$, or what is the same by the $(\leqslant\lambda_\varepsilon)$ that $\Omega, x : G_1 \sqsubseteq G'_1 \vdash t'_1 \leqslant t'_2 : G_2 \leqslant G'_2$, but the result follows immediately by the induction hypothesis on $\Omega, x : G_1 \sqsubseteq G'_1 \vdash \Xi_1 \triangleright t_1 : G_2 \leqslant \Xi_2 \triangleright t_2 : G'_2$, with the translations t'_1 and $t'_2(\Omega, x : G_1 \sqsubseteq G'_1 \equiv \Gamma_1, x : G_1 \sqsubseteq \Gamma_2, x : G'_1)$.

- Case $(\Omega \vdash (\Lambda X.t_1) : \forall X.G_1 \leqslant_v (\Lambda X.t_2) : \forall X.G_2)$. We know that

$$(\leqslant\Lambda)\frac{\Omega \vdash \Xi_1 \triangleright t_1 : G_1 \leqslant \Xi_2 \triangleright t_2 : G_2}{\Omega \vdash (\Lambda X.t_1) : \forall X.G_1 \leqslant_v (\Lambda X.t_2) : \forall X.G_2}$$

$$(\mathrm{G}\lambda)\frac{\Xi_1; \Delta, X; \Gamma_1 \vdash t_1 \rightsquigarrow t'_1 : G_1}{\Xi_1; \Delta; \Gamma_1 \vdash (\Lambda X.t_1) \rightsquigarrow (\Lambda X.t'_1) : \forall X.G_1}$$

$$(\mathrm{G}\lambda)\frac{\Xi_2; \Delta, X; \Gamma_2 \vdash t_2 \rightsquigarrow t'_2 : G_2}{\Xi_2; \Delta; \Gamma_2 \vdash (\Lambda X.t_2) \rightsquigarrow (\Lambda X.t'_2) : \forall X.G_2}$$

Therefore, we are required to prove that $\Omega \vdash (\Lambda X.t'_1) \leqslant (\Lambda X.t'_2) : \forall X.G_1 \leqslant \forall X.G_2$, or what is the same by the rule $(\leqslant\Lambda_\varepsilon)$ that $\Omega \vdash t'_1 \leqslant t'_2 : G_1 \leqslant G_2$, but the result follows immediately by the induction hypothesis on $\Omega \vdash \Xi_1 \triangleright t_1 : G_1 \leqslant \Xi_2 \triangleright t_2 : G_2$, with the translations t'_1 and t'_2.

Case $(\Omega \vdash \Xi_1 \triangleright t_1 :: G_1^* : G_1^* \leqslant \Xi_2 \triangleright t_2 :: G_2^* : G_2^*)$.

$$(\leqslant\mathrm{asct})\frac{\Omega \vdash \Xi_1 \triangleright t_1 : G_1 \leqslant \Xi_2 \triangleright t_2 : G_2 \qquad G_1 \sqcap G_1^* \leqslant G_2 \sqcap G_2^* \qquad G_1^* \sqsubseteq G_2^*}{\Omega \vdash \Xi_1 \triangleright t_1 :: G_1^* : G_1^* \leqslant \Xi_2 \triangleright t_2 :: G_2^* : G_2^*}$$

$$(\mathrm{Gasct})\frac{\Xi_1; \Delta; \Gamma_1 \vdash t_1 \rightsquigarrow t'_1 : G_1 \qquad \varepsilon_1 = \mathcal{I}_\Xi(G_1, G_1^*)}{\Xi_1; \Delta; \Gamma_1 \vdash t_1 :: G_1^* \rightsquigarrow \varepsilon_1 t'_1 :: G_1^* : G_1^*}$$

$$(\mathrm{Gasct})\frac{\Xi_2; \Delta; \Gamma_2 \vdash t_2 \rightsquigarrow t'_2 : G_2 \qquad \varepsilon_2 = \mathcal{I}_\Xi(G_2, G_2^*)}{\Xi_2; \Delta; \Gamma_2 \vdash t_2 :: G_2^* \rightsquigarrow \varepsilon_2 t'_2 :: G_2^* : G_2^*}$$

We have to prove that $\Omega \vdash \varepsilon_1 t'_1 :: G_1^* \leqslant \varepsilon_2 t'_2 :: G_2^* : G_1^* \leqslant G_2^*$, or what is the same by the rule $(\leqslant\mathrm{asc}_\varepsilon)$, we have to prove that $\varepsilon_1 \leqslant \varepsilon_2$, $\Omega \vdash t'_1 \leqslant t'_2 : G_1 \leqslant G_2$ and $G_1^* \sqsubseteq G_2^*$. By Proposition 5.21, we know that $\varepsilon_1 = \mathcal{I}_\Xi(G_1, G_1^*) = \mathcal{I}_\Xi(G_1 \sqcap G_1^*, G_1 \sqcap G_1^*)$ and $\varepsilon_2 = \mathcal{I}_\Xi(G_2, G_2^*) = \mathcal{I}_\Xi(G_2 \sqcap G_2^*, G_2 \sqcap G_2^*)$. Since $G_1 \sqcap G_1^* \leqslant G_2 \sqcap G_2^*$, then $\varepsilon_1 = \mathcal{I}_\Xi(G_1, G_1^*) = \mathcal{I}_\Xi(G_1 \sqcap G_1^*, G_1 \sqcap G_1^*) \leqslant \mathcal{I}_\Xi(G_2 \sqcap G_2^*, G_2 \sqcap G_2^*) = \mathcal{I}_\Xi(G_2, G_2^*) = \varepsilon_2$, by Proposition 5.23. Thus, we only have to prove that $\Omega \vdash t'_1 \leqslant t'_2 : G_1 \leqslant G_2$, and we know that $\Omega \vdash t'_1 : G_1 \leqslant_v t'_2 : G_2$, then by the induction hypothesis the result holds.

Case $(\Omega \vdash \Xi_1 \triangleright t_1\ t_1' : cod^\sharp(G_1) \leqslant \Xi_2 \triangleright t_2\ t_2' : cod^\sharp(G_2))$.

$$(\leqslant\text{app})\frac{\Omega \vdash \Xi_1 \triangleright t_1 : G_1 \leqslant \Xi_2 \triangleright t_2 : G_2 \qquad \Omega \vdash \Xi_1 \triangleright t_1' : G_1' \leqslant \Xi_2 \triangleright t_2' : G_2' \qquad G_1' \sqcap dom^\sharp(G_1) \leqslant G_2' \sqcap dom^\sharp(G_2)}{\Omega \vdash \Xi_1 \triangleright t_1\ t_1' : cod^\sharp(G_1) \leqslant \Xi_2 \triangleright t_2\ t_2' : cod^\sharp(G_2)}$$

$$(\text{Gapp})\frac{\begin{array}{ll}\Xi_1; \Delta; \Gamma_1 \vdash t_1 \rightsquigarrow t_{11} : G_1 & t_{11}' = norm(t_{11}, G_1, dom^\sharp(G_1) \to cod^\sharp(G_1)) \\ \Xi_1; \Delta; \Gamma_1 \vdash t_1' \rightsquigarrow t_{12} : G_1' & t_{12}' = norm(t_{12}, G_1', dom^\sharp(G_1))\end{array}}{\Xi_1; \Delta; \Gamma_1 \vdash t_1\ t_1' \rightsquigarrow t_{11}'\ t_{12}' : cod^\sharp(G_1)}$$

$$(\text{Gapp})\frac{\begin{array}{ll}\Xi_2; \Delta; \Gamma_2 \vdash t_2 \rightsquigarrow t_{21} : G_2 & t_{21}' = norm(t_{21}, G_2, dom^\sharp(G_2) \to cod^\sharp(G_2)) \\ \Xi_2; \Delta; \Gamma_2 \vdash t_2' \rightsquigarrow t_{22} : G_2' & t_{22}' = norm(t_{22}, G_2', dom^\sharp(G_2))\end{array}}{\Xi_2; \Delta; \Gamma_2 \vdash t_2\ t_2' \rightsquigarrow t_{21}'\ t_{22}' : cod^\sharp(G_2)}$$

We have to prove that $\Omega \vdash t_{11}'\ t_{12}' \leqslant t_{21}'\ t_{22}' : cod^\sharp(G_1) \leqslant cod^\sharp(G_2)$, or what is the same by the rule ($\leqslant$app$_\varepsilon$), we have to prove that $\Omega \vdash t_{11}' \leqslant t_{21}' : dom^\sharp(G_1) \to cod^\sharp(G_1) \leqslant dom^\sharp(G_2) \to cod^\sharp(G_2)$ and $\Omega \vdash t_{12}' \leqslant t_{22}' : dom^\sharp(G_1) \leqslant dom^\sharp(G_2)$. We know that

$$t_{11}' = norm(t_{11}, G_1, dom^\sharp(G_1) \to cod^\sharp(G_1)) = \varepsilon_{11} t_{11} :: dom^\sharp(G_1) \to cod^\sharp(G_1)$$

where $\varepsilon_{11} = \mathcal{I}_{\Xi_1}(G_1, dom^\sharp(G_1) \to cod^\sharp(G_1)) = \mathcal{I}_{\Xi_1}(dom^\sharp(G_1) \to cod^\sharp(G_1), dom^\sharp(G_1) \to cod^\sharp(G_1)) = \varepsilon_{dom^\sharp(G_1) \to cod^\sharp(G_1)}$

$$t_{21}' = norm(t_{21}, G_2, dom^\sharp(G_2) \to cod^\sharp(G_2)) = \varepsilon_{21} t_{21} :: dom^\sharp(G_2) \to cod^\sharp(G_2)$$

where $\varepsilon_{21} = \mathcal{I}_{\Xi_2}(G_2, dom^\sharp(G_2) \to cod^\sharp(G_2)) = \mathcal{I}_{\Xi_2}(dom^\sharp(G_2) \to cod^\sharp(G_2), dom^\sharp(G_2) \to cod^\sharp(G_2)) = \varepsilon_{dom^\sharp(G_2) \to cod^\sharp(G_2)}$

By induction hypothesis on $\Omega \vdash \Xi_1 \triangleright t_1 : G_1 \leqslant \Xi_2 \triangleright t_2 : G_2$, we know that $\Omega \vdash t_{11} \leqslant t_{21} : G_1 \leqslant G_2$, and by Proposition 5.22, we know that $G_1 \sqsubseteq G_2$, thus $dom^\sharp(G_1) \to cod^\sharp(G_1) \sqsubseteq dom^\sharp(G_2) \to cod^\sharp(G_2)$. Therefore, we only have to prove by rule ($\leqslant$Masc$_\varepsilon$) that $\varepsilon_{11} \sqsubseteq \varepsilon_{21}$. But, by Proposition 5.24 and $dom^\sharp(G_1) \to cod^\sharp(G_1) \sqsubseteq dom^\sharp(G_2) \to cod^\sharp(G_2)$ the results holds.

Also, we know that

$$t_{12}' = norm(t_{12}, G_1', dom^\sharp(G_1)) = \varepsilon_{12} t_{12} :: dom^\sharp(G_1) \text{ where } \varepsilon_{12} = \mathcal{I}_{\Xi_1}(G_1', dom^\sharp(G_1))$$

$$t_{22}' = norm(t_{22}, G_2', dom^\sharp(G_2)) = \varepsilon_{22} t_{22} :: dom^\sharp(G_2) \text{ where } \varepsilon_{22} = \mathcal{I}_{\Xi_2}(G_2', dom^\sharp(G_2))$$

By induction hypothesis on $\Omega \vdash \Xi_1 \triangleright t_1' : G_1' \leqslant \Xi_2 \triangleright t_2' : G_2'$, we know that $\Omega \vdash t_{12} \leqslant t_{22} : G_1' \leqslant G_2'$. and and by Proposition 5.22, we know that $dom^\sharp(G_1) \sqsubseteq dom^\sharp(G_2)$. By Proposition 5.23 and $G_1' \sqcap dom^\sharp(G_1) \leqslant G_2' \sqcap dom^\sharp(G_2)$, we know that

$$\varepsilon_{12} = \mathcal{I}_{\Xi_1}(G_1', dom^\sharp(G_1)) = \mathcal{I}_{\Xi_1}(G_1' \sqcap dom^\sharp(G_1), G_1' \sqcap dom^\sharp(G_1)) \leqslant$$
$$\mathcal{I}_{\Xi_2}(G_2' \sqcap dom^\sharp(G_2), G_2' \sqcap dom^\sharp(G_2)) = \mathcal{I}_{\Xi_2}(G_2', dom^\sharp(G_2)) = \varepsilon_{22}$$

Therefore, the results holds.

Case $(\Omega \vdash \Xi_1 \triangleright t_1\ [G_1'] : inst^\sharp(G_1, G_1') \leqslant \Xi_2 \triangleright t_2\ [G_2'] : inst^\sharp(G_2, G_2'))$.

$$(\leqslant\text{appG})\frac{\Omega \vdash \Xi_1 \triangleright t_1 : G_1 \leqslant \Xi_2 \triangleright t_2 : G_2 \qquad G_1' \leqslant G_2'}{\Omega \vdash \Xi_1 \triangleright t_1\ [G_1'] : inst^\sharp(G_1, G_1') \leqslant \Xi_2 \triangleright t_2\ [G_2'] : inst^\sharp(G_2, G_2')}$$

$$\text{(GappG)}\frac{\Xi_1;\Delta;\Gamma_1 \vdash t_1 \rightsquigarrow t_1' : G_1 \qquad t_1'' = norm(t_1', G_1, \forall var^\sharp(G_1).schm_u^\sharp(G_1))}{\Xi_1;\Delta;\Gamma_1 \vdash t_1\ [G_1'] \rightsquigarrow t_1''\ [G_1'] : inst^\sharp(G_1, G_1')}$$

$$\text{(GappG)}\frac{\Xi_2;\Delta;\Gamma_2 \vdash t_2 \rightsquigarrow t_2' : G_2 \qquad t_2'' = norm(t_2', G_2, \forall var^\sharp(G_2).schm_u^\sharp(G_2))}{\Xi_2;\Delta;\Gamma_2 \vdash t_2\ [G_2'] \rightsquigarrow t_2''\ [G_2'] : inst^\sharp(G_2, G_2')}$$

We have to prove that $\Omega \vdash t_1''\ [G_1'] \leqslant t_2''\ [G_2'] : G_1^* \leqslant G_2^*$, or what is the same by the rule ($\leqslant$appG$_\varepsilon$), we have to prove that $t_1'' \leqslant t_2''$ and $G_1' \leqslant G_2'$. $G_1' \leqslant G_2'$ follows by premise. We know that

$$t_1'' = norm(t_1', G_1, \forall var^\sharp(G_1).schm_u^\sharp(G_1)) = \varepsilon_1 t_1' :: \forall var^\sharp(G_1).schm_u^\sharp(G_1)$$

where $\varepsilon_1 = \mathcal{I}_{\Xi_1}(G_1, \forall var^\sharp(G_1).schm_u^\sharp(G_1)) = \mathcal{I}_{\Xi_1}(\forall var^\sharp(G_1).schm_u^\sharp(G_1), \forall var^\sharp(G_1).schm_u^\sharp(G_1)) = \varepsilon_{\forall var^\sharp(G_1).schm_u^\sharp(G_1)}$

$$t_2'' = norm(t_2', G_2, \forall var^\sharp(G_2).schm_u^\sharp(G_2)) = \varepsilon_2 t_2' :: \forall var^\sharp(G_2).schm_u^\sharp(G_2)$$

where $\varepsilon_2 = \mathcal{I}_{\Xi_2}(G_2, \forall var^\sharp(G_2).schm_u^\sharp(G_2)) = \mathcal{I}_{\Xi_2}(\forall var^\sharp(G_2).schm_u^\sharp(G_2), \forall var^\sharp(G_2).schm_u^\sharp(G_2)) = \varepsilon_{\forall var^\sharp(G_2).schm_u^\sharp(G_2)}$

By induction hypothesis on $\Omega \vdash \Xi_1 \triangleright t_1 : G_1 \leqslant \Xi_2 \triangleright t_2 : G_2$, we know that $\Omega \vdash t_1' \leqslant t_2' : G_1 \leqslant G_2$, and by Proposition 5.22, we know that $G_1 \sqsubseteq G_2$, thus $\forall var^\sharp(G_1).schm_u^\sharp(G_1) \sqsubseteq \forall var^\sharp(G_2).schm_u^\sharp(G_2)$. Therefore, we only have to prove by rule ($\leqslant$Masc$_\varepsilon$) that $\varepsilon_1 \sqsubseteq \varepsilon_2$. But, by Proposition 5.24 and $\forall var^\sharp(G_1).schm_u^\sharp(G_1) \sqsubseteq \forall var^\sharp(G_2).schm_u^\sharp(G_2)$ the results holds.

$\square$

6 GSF: PARAMETRICITY

In this section we present the logical relation for parametricity of GSF, the proof of the fundamental property, and the soundness of the logical relation wrt contextual approximation.

6.1 Auxiliary Definitions

In this section we show function definitions used in the logical relation of GSF (Figure 12).

Definition 6.1. $ev(\varepsilon u :: G) = \varepsilon$

Definition 6.2.

$$const(E) = \begin{cases} B & E = B \\ ? \rightarrow ? & E = E_1 \rightarrow E_2 \\ \forall X.? & E = \forall X.E_1 \\ ? \times ? & E = E_1 \times E_2 \\ \alpha & E = \alpha^{E_1} \\ X & E = X \\ ? & E = ? \end{cases}$$

6.2 Fundamental Property

Theorem 10.1 (Fundamental Property). *If* $\Xi; \Delta; \Gamma \vdash t : G$ *then* $\Xi; \Delta; \Gamma \vdash t \preceq t : G$.

Proof. By induction on the type derivation of t.

Case (Easc). Then $t = \varepsilon s :: G$, and therefore:

$$\text{(Easc)}\frac{\Xi; \Delta; \Gamma \vdash s : G' \qquad \boxed{\varepsilon \Vdash \Xi; \Delta \vdash G' \sim G}}{\Xi; \Delta; \Gamma \vdash \boxed{\varepsilon s :: G} : G}$$

We follow by induction on the structure of s.

- If $s = b$ then:

$$\text{(Eb)}\frac{ty(b) = B \qquad \Xi; \Delta \vdash \Gamma}{\Xi; \Delta; \Gamma \vdash b : B}$$

Then we have to prove that $\Xi; \Delta; \Gamma \vdash \varepsilon b :: G \preceq \varepsilon b :: G : G$, but the result follows directly by Prop 6.3 (Compatibility of Constant).

- If $s = \lambda x : G_1.t'$ then:

$$\text{(E}\lambda\text{)}\frac{\Xi; \Delta; \Gamma, x : G_1 \vdash t' : G_2}{\Xi; \Delta; \Gamma \vdash \lambda x : G_1.t' : G_1 \rightarrow G_2}$$

Then we have to prove that:

$$\Xi; \Delta; \Gamma \vdash \varepsilon(\lambda x : G_1.t') :: G \preceq \varepsilon(\lambda x : G_1.t') :: G : G$$

By induction hypotheses we already know that $\Xi; \Delta; \Gamma, x : G_1 \vdash t' \preceq t' : G_2$. But the result follows directly by Prop 6.4 (Compatibility of term abstraction).

- If $s = \Lambda X.t'$ then:

$$\text{(E}\Lambda\text{)}\frac{\Xi; \Delta, X; \Gamma \vdash t' : G^* \qquad \Xi; \Delta \vdash \Gamma}{\Xi; \Delta; \Gamma \vdash \Lambda X.t' : \forall X.G^*}$$

Then we have to prove that:

$$\Xi; \Delta; \Gamma \vdash \varepsilon(\Lambda X.t') :: G \preceq \varepsilon(\Lambda X.t') :: G : G$$

By induction hypotheses we already know that $\Xi; \Delta, X; \Gamma \vdash t' \preceq t' : G^*$. But the result follows directly by Prop 10.2 (Compatibility of type abstraction).

- If $s = \langle u_1, u_2 \rangle$ then:

$$\text{(Epair)}\frac{\Xi;\Delta;\Gamma \vdash u_1 : G_1 \qquad \Xi;\Delta;\Gamma \vdash u_2 : G_2}{\Xi;\Delta;\Gamma \vdash \langle u_1, u_2 \rangle : G_1 \times G_2}$$

Then we have to prove that:

$$\Xi;\Delta;\Gamma \vdash \varepsilon\langle u_1, u_2 \rangle :: G \leq \varepsilon\langle u_1, u_2 \rangle :: G : G$$

We know by premise that $\Xi;\Delta;\Gamma \vdash \pi_1(\varepsilon)u_1 :: G_1 : G_1$ and $\Xi;\Delta;\Gamma \vdash \pi_2(\varepsilon)u_2 :: G_2 : G_2$. Then by induction hypotheses we already know that: $\Xi;\Delta;\Gamma \vdash \pi_1(\varepsilon)u_1 :: G_1 \leq \pi_1(\varepsilon)u_1 :: G_1 : G_1$ and $\Xi;\Delta;\Gamma \vdash \pi_2(\varepsilon)u_2 :: G_2 \leq \pi_2(\varepsilon)u_2 :: G_2 : G_2$. But the result follows directly by Prop 6.5 (Compatibility of pairs).

- If $s = t'$, and therefore:

$$\text{(Easc)}\frac{\Xi;\Delta;\Gamma \vdash t' : G' \qquad \varepsilon \vdash \Xi;\Delta \vdash G' \sim G}{\Xi;\Delta;\Gamma \vdash \varepsilon t' :: G : G}$$

By induction hypotheses we already know that $\Xi;\Delta;\Gamma \vdash t' \leq t' : G'$, then the result follows directly by Prop 6.8 (Compatibility of ascriptions).

Case (Epair). Then $t = \langle t_1, t_2 \rangle$, and therefore:

$$\text{(Epair)}\frac{\Xi;\Delta;\Gamma \vdash t_1 : G_1 \qquad \Xi;\Delta;\Gamma \vdash t_2 : G_2}{\Xi;\Delta;\Gamma \vdash \langle t_1, t_2 \rangle : G_1 \times G_2}$$

where $G = G_1 \times G_2$ Then we have to prove that:

$$\Xi;\Delta;\Gamma \vdash \langle t_1, t_2 \rangle \leq \langle t_1, t_2 \rangle : G_1 \times G_2$$

By induction hypotheses we already know that: $\Xi;\Delta;\Gamma \vdash t_1 \leq t_1 : G_1$ and $\Xi;\Delta;\Gamma \vdash t_2 \leq t_2 : G_2$. But the result follows directly by Prop 6.6 (Compatibility of pairs).

Case (Ex). Then $t = x$, and therefore:

$$\text{(Ex)}\frac{x : G \in \Gamma \qquad \Xi;\Delta \vdash \Gamma}{\Xi;\Delta;\Gamma \vdash x : G}$$

Then we have to prove that $\Xi;\Delta;\Gamma \vdash x \leq x : G$. But the result follows directly by Prop 6.7 (Compatibility of variables).

Case (Eop). Then $t = op(\overline{t'})$, and therefore:

$$\text{(Eop)}\frac{\Xi;\Delta;\Gamma \vdash \overline{t'} : \overline{G'} \qquad ty(op) = \overline{G'} \to G}{\Xi;\Delta;\Gamma \vdash op(\overline{t'}) : G}$$

Then we have to prove that: $\Xi;\Delta;\Gamma \vdash op(\overline{t'}) \leq op(\overline{t'}) : G$. By the induction hypothesis we obtain that: $\Xi;\Delta;\Gamma \vdash \overline{t'} \leq \overline{t'} : \overline{G}$. Then the result follows directly by Prop 6.9 (Compatibility of app operator).

Case (Eapp). Then $t = t_1\ t_2$, and therefore:

$$\text{(Eapp)}\frac{\Xi;\Delta;\Gamma \vdash t_1 : G_{11} \to G_{12} \qquad \Xi;\Delta;\Gamma \vdash t_2 : G_{11}}{\Xi;\Delta;\Gamma \vdash t_1\ t_2 : G_{12}}$$

where $G = G_{12}$. Then we have to prove that:

$$\Xi;\Delta;\Gamma \vdash t_1\ t_2 \leq t_1\ t_2 : G_{12}$$

By the induction hypothesis we obtain that: $\Xi;\Delta;\Gamma \vdash t_1 \leq t_1 : G_{11} \to G_{12}$ and $\Xi;\Delta;\Gamma \vdash t_2 \leq t_2 : G_{11}$. Then the result follows directly by Prop 6.10 (Compatibility of term application).

Case (EappG). Then $t = t' \; [G_2]$, and therefore:

$$\text{(EappG)}\frac{\Xi;\Delta;\Gamma \vdash t' : \forall X.G_1 \qquad \Xi;\Delta \vdash G_2}{\Xi;\Delta;\Gamma \vdash t' \; [G_2] : G_1[G_2/X]}$$

where $G = G_1[G_2/X]$. Then we have to prove that:

$$\Xi;\Delta;\Gamma \vdash t' \; [G_2] \leq t' \; [G_2] : G_1[G_2/X]$$

By induction hypotheses we know that:

$$\Xi;\Delta;\Gamma \vdash t' \leq t' : \forall X.G_1$$

Then the result follows directly by Prop 10.3 (Compatibility of type application).

Case (Epair1). Then $t = \pi_1(t')$, and therefore:

$$\text{(Epair1)}\frac{\Xi;\Delta;\Gamma \vdash t' : G_1 \times G_2}{\Xi;\Delta;\Gamma \vdash \pi_1(t') : G_1}$$

where $G = G_1$. Then we have to prove that: $\Xi;\Delta;\Gamma \vdash \pi_1(t') \leq \pi_1(t') : G_1$. By the induction hypothesis we obtain that: $\Xi;\Delta;\Gamma \vdash t' \leq t' : G_1 \times G_2$. Then the result follows directly by Prop 6.11 (Compatibility of access to the first component of the pair).

Case (Epair2). Then $t = \pi_2(t')$, and therefore:

$$\text{(Epair2)}\frac{\Xi;\Delta;\Gamma \vdash t' : G_1 \times G_2}{\Xi;\Delta;\Gamma \vdash \pi_2(t') : G_2}$$

where $G = G_2$. Then we have to prove that: $\Xi;\Delta;\Gamma \vdash \pi_2(t') \leq \pi_2(t') : G_2$. By the induction hypothesis we obtain that: $\Xi;\Delta;\Gamma \vdash t' \leq t' : G_1 \times G_2$. Then the result follows directly by Prop 6.12 (Compatibility of access to the second component of the pair).

$\square$

In order to prove parametricity, we add an index to the evidence and we are more detailed in the reduction rules. A brief explanation is given below. The index of an evidence is an integer greater than cero. To know the index of an evidence ε, we use the following operator $\varepsilon.n = k$, which specifies that the index of the evidence ε is the integer $k > 0$. The reduction rules always took a step. Here we redefine them and they can take one or more steps. This will depend on whether or not a transitivity of evidence is applied. If it does, the rule will take as many steps as the evidence index on the right. Below we define the steps in the rules

$\boxed{\Xi \triangleright t \longrightarrow \Xi \triangleright t \text{ or } \mathbf{error}}$ **Notion of reduction**

$$(R\text{asc}) \quad \Xi \triangleright \varepsilon_2(\varepsilon_1 u :: G_1) :: G_2 \xrightarrow{k} \begin{cases} \Xi \triangleright (\varepsilon_1 \mathbin{⨾} \varepsilon_2)u :: G_2 & \text{if } \varepsilon_2.n = k \\ \mathbf{error} & \text{if not defined} \end{cases}$$

$$(R\text{op}) \quad \Xi \triangleright op(\overline{\varepsilon u :: G}) \xrightarrow{1} \Xi \triangleright \varepsilon_B \; \delta(op, \overline{u}) :: B \quad \text{where } B \triangleq cod(ty(op))$$

$$(R\text{app}) \quad \Xi \triangleright (\varepsilon_1(\lambda x : G_{11}.t) :: G_1 \rightarrow G_2) \; (\varepsilon_2 u :: G_1) \xrightarrow{k+1} \begin{cases} \Xi \triangleright cod(\varepsilon_1)(t[((\varepsilon_2 \mathbin{⨾} dom(\varepsilon_1))u :: G_{11})/x]) :: G_2 \\ \quad \text{if } dom(\varepsilon_1) = k \\ \mathbf{error} \quad \text{if not defined} \end{cases}$$

$$(R\text{pair}) \quad \Xi \triangleright \langle \varepsilon_1 u_1 :: G_1, \varepsilon_2 u_2 :: G_2 \rangle \xrightarrow{1} \Xi \triangleright (\varepsilon_1 \times \varepsilon_2)\langle u_1, u_2 \rangle :: G_1 \times G_2$$

$$(R\text{proj}i) \quad \Xi \triangleright \pi_i(\varepsilon \langle u_1, u_2 \rangle :: G_1 \times G_2) \xrightarrow{1} \Xi \triangleright p_i(\varepsilon) u_i :: G_i$$

$$(R\text{appG}) \quad \Xi \triangleright (\varepsilon \Lambda X.t :: \forall X.G) \; [G'] \xrightarrow{1} \Xi' \triangleright \varepsilon_{out}(\varepsilon[\hat{\alpha}]t[\hat{\alpha}/X] :: G[\alpha/X]) :: G[G'/X]$$

where $\Xi' \triangleq \Xi, \alpha := G'$ for some $\alpha \notin dom(\Xi)$ and $\hat{\alpha} = lift_{\Xi'}(\alpha)$

Proposition 6.3 (**Compatibility-Eb**). *If $b \in B$, $\varepsilon \vdash \Xi; \Delta \vdash B \sim G$ and $\Xi; \Delta \vdash \Gamma$ then:*

$$\Xi; \Delta; \Gamma \vdash \varepsilon b :: G \preceq \varepsilon b :: G : G$$

Proof. As b is constant then it does not have free variables or type variables, then $b = \rho(\gamma_i(b))$. Then we have to prove that for all $W \in \mathcal{S}[\![\Xi]\!]$ it is true that:

$$(W, \rho_1(\varepsilon)b :: \rho(G), \rho_2(\varepsilon)b :: \rho(G) \in \mathcal{T}_\rho[\![G]\!]$$

As $\rho_i(\varepsilon)b :: G$ are values, then we have to prove that:

$$(W, \rho_1(\varepsilon)b :: \rho(G), \rho_2(\varepsilon)b :: \rho(G)) \in \mathcal{V}_\rho[\![G]\!]$$

(1) $G = B$, we know that $\langle B, B\rangle = \varepsilon \vdash \Xi; \Delta \vdash B \sim B$, then $\rho_i(\varepsilon) = \varepsilon$ and the result follows immediately by the definition of $\mathcal{V}_\rho[\![B]\!]$.

(2) If $G \in$ TypeName then $\varepsilon = \langle H_3, \alpha^{E_4}\rangle$. Notice that as α^{E_4} cannot have free type variables therefore H_3 neither. Then $\varepsilon = \rho_i(\varepsilon)$. As α is sync, then let us call $G'' = W.\Xi_i(\alpha)$. We have to prove that:

$$(W, \langle H_3, \alpha^{E_4}\rangle\, b :: \alpha, \langle H_3, \alpha^{E_4}\rangle\, b :: \alpha) \in \mathcal{V}_\rho[\![\alpha]\!]$$

which, by definition of $\mathcal{V}_\rho[\![\alpha]\!]$, is equivalent to prove that:

$$(\downarrow W, \langle H_3, E_4\rangle\, b :: G'', \langle E_3, E_4\rangle\, b :: G'') \in \mathcal{V}_\rho[\![G'']\!]$$

Then we proceed by case analysis on ε:

- (Case $\varepsilon = \langle H_3, \alpha^{\beta^{E_4}}\rangle$). We know that $\langle H_3, \alpha^{\beta^{E_4}}\rangle \vdash \Xi; \Delta \vdash B \sim \alpha$, then by Lemma 6.29, $\langle H_3, \beta^{E_4}\rangle \vdash \Xi; \Delta \vdash B \sim G''$. As $\beta^{E_4} \sqsubseteq G''$, then G'' can either be ? or β.
  If $G'' = ?$, then by definition of $\mathcal{V}_\rho[\![?]\!]$, we have to prove that the resulting values belong to $\mathcal{V}_\rho[\![\beta]\!]$. Also as $\langle H_3, \beta^{E_4}\rangle \vdash \Xi; \Delta \vdash B \sim ?$, by Lemma 6.27, $\langle H_3, \beta^{E_4}\rangle \vdash \Xi; \Delta \vdash B \sim \beta$, and then we proceed just like this case once again (this is process is finite as there are no circular references by construction and it ends up in something different to a type name).
  If $G'' = \beta$ we use an analogous argument as for $G'' = ?$.
- (Case $\varepsilon = \langle H_3, \alpha^{H_4}\rangle$). We have to prove that

$$(\downarrow W, \langle H_3, H_4\rangle\, b :: G'', \langle H_3, H_4\rangle\, b :: G'') \in \mathcal{V}_\rho[\![G'']\!]$$

  By Lemma 6.29, $\langle H_3, H_4\rangle \vdash \Xi; \Delta \vdash B \sim G''$. Then if $G'' = ?$, we proceed as the case $G = ?$, with the evidence $\varepsilon = \langle H_3, H_4\rangle$. If $G'' \in$ HeadType, we proceed as the previous case where $G = B$, and the evidence $\varepsilon = \langle H_3, H_4\rangle$.
  Also, we have to prove that $(\forall \Xi', \varepsilon', G_1^*$, such that $\varepsilon'.n = k$, $\varepsilon' = \langle \alpha^{E_1^{**}}, E_2^{**}\rangle$ $(\downarrow W \in \mathcal{S}[\![\Xi']\!] \wedge \varepsilon' \vdash \Xi' \vdash \alpha \sim G_1^*)$, we get that

$$(\downarrow_1 W, \varepsilon'(\langle H_3, \alpha^{H_4}\rangle u_1 :: \alpha) :: G_1^*, \varepsilon'(\langle H_4, \alpha^{E_{22}}\rangle u_2 :: \alpha) :: G_1^*) \in \mathcal{T}_\rho[\![G_1^*]\!])$$

  or what is the same (($\langle H_3, \alpha^{H_4}\rangle \mathbin{\text{;}} \varepsilon'$) fails the result follows immediately)

$$(\downarrow_{1+k} W, (\langle H_3, \alpha^{H_4}\rangle \mathbin{\text{;}} \varepsilon')u_1 :: G_1^*, (\langle H_2, \alpha^{H_4}\rangle \mathbin{\text{;}} \varepsilon')u_2 :: G_1^*) \in \mathcal{V}_\rho[\![G_1^*]\!])$$

  By definition of transitivity and Lemma 6.30, we know that

$$\langle H_3, \alpha^{H_4}\rangle \mathbin{\text{;}} \langle \alpha^{E_1^{**}}, E_2^{**}\rangle = \langle H_3, H_4\rangle \mathbin{\text{;}} \langle E_1^{**}, E_2^{**}\rangle$$

  We know that $\langle E_1^{**}, E_2^{**}\rangle \vdash \Xi' \vdash G'' \sim G_1^*$. Since $\langle E_1^{**}, E_2^{**}\rangle \vdash \Xi \vdash G'' \sim G_1^*$, $\downarrow_1 W \in \mathcal{S}[\![\Xi']\!]$, $(\downarrow_1 W, \langle H_3, H_4\rangle u_1 :: G'', \langle H_1, H_4\rangle u_2 :: G'') \in \mathcal{V}_\rho[\![G'']\!]$, by Lemma 6.17, we know that (since

$(\langle H_3, \alpha^{H_4}\rangle \circ \varepsilon')$ does not fail then $(\langle H_3, H_4\rangle \circ \langle E_1^{**}, E_2^{**}\rangle)$ also does not fail by the transitivity rules)

$$(\downarrow_{1+k} W, (\langle H_3, H_4\rangle \circ \langle E_1^{**}, E_2^{**}\rangle)u_1 :: G_1^*, (\langle H_3, H_4\rangle \circ \langle E_1^{**}, E_2^{**}\rangle)u_2 :: G_1^*) \in \mathcal{V}_\rho[\![G_1^*]\!])$$

The result follows immediately.

(3) If $G = ?$ we have the following cases:

- $(G = ?, \varepsilon = \langle H_3, H_4\rangle)$. By the definition of $\mathcal{V}_\rho[\![?]\!]$ in this case we have to prove that:

$$(W, \rho_1(\varepsilon)b :: \mathit{const}(H_4), \rho_2(\varepsilon)b :: \mathit{const}(H_4)) \in \mathcal{V}_\rho[\![\mathit{const}(H_4)]\!]$$

but as $\mathit{const}(H_4) = B$ (note that $H_3 = B$ then since $H_4 \in$ HeadType has to be B). The the result follows immediately since is part of the premise.

- $(G = ?, \varepsilon = \langle H_3, \alpha^{E_4}\rangle)$. Notice that as α^{E_4} cannot have free type variables therefore E_3 neither. Then $\varepsilon = \rho_i(\varepsilon)$. By the definition of $\mathcal{V}_\rho[\![?]\!]$ we have to prove that

$$(W, \langle H_3, \alpha^{E_4}\rangle\, u_1 :: \alpha, \langle H_3, \alpha^{E_4}\rangle\, u_2 :: \alpha) \in \mathcal{V}_\rho[\![\alpha]\!]$$

Note that by Lemma 6.27 we know that $\varepsilon \vdash \Xi; \Delta \vdash B \sim \alpha$. Then we proceed just like the case $G \in$ TypeName.

□

Proposition 6.4 (**Compatibility-Eλ**). *If* $\Xi; \Delta; \Gamma, x : G_1 \vdash t \preceq t' : G_2$*,* $\varepsilon \vdash \Xi; \Delta \vdash G_1 \rightarrow G_2 \sim G$ *then:*

$$\Xi; \Delta; \Gamma \vdash \varepsilon(\lambda x : G_1.t) :: G \preceq \varepsilon(\lambda x : G_1.t') :: G : G$$

Proof. First, we are required to show that $\Xi; \Delta; \Gamma \vdash \varepsilon(\lambda x : G_1.t) :: G : G$ and $\Xi; \Delta; \Gamma \vdash \varepsilon(\lambda x : G_1.t') :: G : G$, which follow from $\varepsilon \vdash \Xi; \Delta \vdash G_1 \rightarrow G_2 \sim G$ and $\Xi; \Delta; \Gamma \vdash \lambda x : G_1.t : G_1 \rightarrow G_2$ and $\Xi; \Delta; \Gamma \vdash \lambda x : G_1.t' : G_1 \rightarrow G_2$ respectively, which follow (respectively) from $\Xi; \Delta; \Gamma, x : G_1 \vdash t : G_2$ and $\Xi; \Delta; \Gamma, x : G_1 \vdash t' : G_2$, which follow from $\Xi; \Delta; \Gamma, x : G_1 \vdash t \preceq t' : G_2$.

Consider arbitrary W, ρ, γ such that $W \in \mathcal{S}[\![\Xi]\!]$, $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$ and $(W, \gamma) \in \mathcal{G}_\rho[\![\Gamma]\!]$. We are required to show that:

$$(W, \rho(\gamma_1(\varepsilon(\lambda x : G_1.t) :: G)), \rho(\gamma_2(\varepsilon(\lambda x : G_1.t) :: G))) \in \mathcal{T}_\rho[\![G]\!]$$

Consider arbitrary i, v_1 and Ξ_1 such that $i < W.j$ and:

$$W.\Xi_1 \triangleright \rho(\gamma_1(\varepsilon(\lambda x : G_1.t) :: G)) \longrightarrow^i \Xi_1 \triangleright v_1$$

Since $\rho(\gamma_1(\varepsilon(\lambda x : G_1.t) :: G)) = \varepsilon_1^\rho(\lambda x : \rho(G_1).\rho(\gamma_1(t))) :: \rho(G)$ and $\varepsilon_2^\rho(\lambda x : \rho(G_1).\rho(\gamma_2(t'))) :: \rho(G)$ is already a value, where $\varepsilon_i^\rho = \rho_i(\varepsilon)$, we have $i = 0$ and $v_1 = \varepsilon_1^\rho(\lambda x : \rho(G_1).\rho(\gamma_1(t))) :: \rho(G)$ and $\Xi_1 = W.\Xi_1$. Since $\varepsilon_2^\rho(\lambda x : \rho(G_1).\rho(\gamma_2(t'))) :: \rho(G)$ is already a value, we are required to show that $\exists W'$, such that $W'.j + i = W.j$, $W' \geq W$, $W'.\Xi_1 = \Xi_1$, $W'.\Xi_2 = \Xi_2$ and:

$$(W', \varepsilon_1^\rho(\lambda x : \rho(G_1).\rho(\gamma_1(t))) :: \rho(G), \varepsilon_2^\rho(\lambda x : \rho(G_1).\rho(\gamma_2(t'))) :: \rho(G)) \in \mathcal{V}_\rho[\![G]\!]$$

Let $W' = W$, then we have to show that:

$$(W, \varepsilon_1^\rho(\lambda x : \rho(G_1).\rho(\gamma_1(t))) :: \rho(G), \varepsilon_2^\rho(\lambda x : \rho(G_1).\rho(\gamma_2(t'))) :: \rho(G)) \in \mathcal{V}_\rho[\![G]\!]$$

Let's suppose that $\varepsilon_1^\rho.n = k$.

First we have to prove that:

$$W.\Xi_1; \Delta; \Gamma \vdash \varepsilon_1^\rho(\lambda x : \rho(G_1).\rho(\gamma_1(t))) :: \rho(G) : \rho(G)$$

As we know that $\Xi; \Delta; \Gamma \vdash \varepsilon(\lambda x : G_1.t) :: G : G$, by Lemma 6.25 the result follows immediately. The case $W.\Xi_2; \Delta; \Gamma \vdash \varepsilon_2^\rho(\lambda x : \rho(G_1).\rho(\gamma_2(t'))) :: \rho(G) : \rho(G)$ is similar.

The type G can be $G_1' \rightarrow G_2'$, for some G_1' and G_2', or ? or a TypeName.

(1) $G = G'_1 \rightarrow G'_2$, we are required to show that $\forall W'', v'_1 = \varepsilon'_1 u'_1 :: \rho(G'_1), v'_2 = \varepsilon'_2 u'_2 :: \rho(G'_1)$, such that $W'' \geq W$ and $(\downarrow W'', v'_1, v'_2) \in \mathcal{V}_\rho[\![G'_1]\!]$, it is true that:

$$(W'', \varepsilon^\rho_1(\lambda x : \rho(G_1).\rho(\gamma_1(t))) :: \rho(G'_1 \rightarrow G'_2)\; v'_1, \varepsilon^\rho_2(\lambda x : \rho(G_1).\rho(\gamma_2(t'))) :: \rho(G'_1 \rightarrow G'_2)\; v'_2) \in \mathcal{T}_\rho[\![G'_2]\!]$$

If $(\varepsilon'_1 \,{}^\circ_9\, dom(\varepsilon^\rho_1))$ fails, then by Lemma 6.26 $(\varepsilon'_2 \,{}^\circ_9\, dom(\varepsilon^\rho_2))$ and the result follows immediately. Else, if $(\varepsilon'_i \,{}^\circ_9\, dom(\varepsilon^\rho_i))$ follows, where $dom(\varepsilon^\rho_1).n = k$, we know that

$$W''.\Xi_1 \triangleright \varepsilon^\rho_1(\lambda x : \rho(G_1).\rho(\gamma_1(t))) :: \rho(G'_1 \rightarrow G'_2)\; v'_1 \longrightarrow^{k+1}$$

$$W''.\Xi_1 \triangleright cod(\varepsilon^\rho_1)(\rho(\gamma_1(t))[(\varepsilon'_1 \,{}^\circ_9\, dom(\varepsilon^\rho_1)u'_1 :: \rho(G_1))/x]) :: \rho(G'_2)\; v'_1 \longrightarrow^{k^*}$$
$$\Xi_1 \triangleright cod(\varepsilon^\rho_1)v_{1f} :: \rho(G'_2) \longrightarrow k$$
$$\Xi_1 \triangleright v^*_1$$

Thus, we have to prove that there exists W^*, such that:

$$W''.\Xi_2 \triangleright \varepsilon^\rho_2(\lambda x : \rho(G_1).\rho(\gamma_2(t'))) :: \rho(G'_1 \rightarrow G'_2)\; v'_2 \longrightarrow^* \Xi_2 \triangleright v^*_2$$

and $(W^*, v^*_1, v^*_2) \in \mathcal{V}_\rho[\![G'_2]\!]$, $W^*.j + 1 + 2k + k^* = W''.j$, $W^*.\Xi_1 = \Xi_1$ and $W^*.\Xi_2 = \Xi_2$.
Note that $dom(\varepsilon^\rho_i) \vdash W''.\Xi_i \vdash \rho(G'_1) \sim \rho(G_1)$. By the Lemma 6.17 (with the type G_1 and the evidences $dom(\varepsilon^\rho_i) \vdash W''.\Xi_i \vdash \rho(G'_1) \sim \rho(G_1)$) it is true that:

$$(\downarrow_1 W'', dom(\varepsilon^\rho_1)v'_1 :: G_1, dom(\varepsilon^\rho_2)v'_2 :: G_1) \in \mathcal{T}_\rho[\![G_1]\!]$$

Since $(\varepsilon'_i \,{}^\circ_9\, dom(\varepsilon^\rho_i))$ does not fail, it is true that:

$$((\downarrow_{k+1} W''), (\varepsilon'_1 \,{}^\circ_9\, dom(\varepsilon^\rho_1))u'_1 :: G_1, (\varepsilon'_2 \,{}^\circ_9\, dom(\varepsilon^\rho_2))u'_2 :: G_1) \in \mathcal{V}_\rho[\![G_1]\!]$$

We instantiate the hypothesis $\Xi; \Delta; \Gamma \vdash t \preceq t' : G_2$, with $(\downarrow_{k+1} W'')$, ρ and $\gamma[x : \rho(G_1) \mapsto (v''_1, v''_2)]$, where $v''_i = (\varepsilon'_i \,{}^\circ_9\, dom(\varepsilon^\rho_i))u'_i :: \rho(G_1)$. Note that $\mathcal{S}[\![\Xi]\!] \ni (\downarrow_{k+1} W'') \geq W$ by the definition of $\mathcal{S}[\![\Xi]\!]$, $((\downarrow_{k+1} W''), \rho) \in \mathcal{D}[\![\Delta]\!]$ by the definition of $\mathcal{D}[\![\Delta]\!]$ and $((\downarrow_{k+1} W''), \gamma[x \mapsto (v''_1, v''_2)]) \in \mathcal{G}_\rho[\![\Gamma, x : \rho(G_1)]\!]$, which follow from: $((\downarrow_{k+1} W''), \gamma) \in \mathcal{G}_\rho[\![\Gamma]\!]$ and $((\downarrow_{k+1} W''), v''_1, v''_2) \in \mathcal{V}_\rho[\![G_1]\!]$ which follows from above. Then, we have that:

$$((\downarrow_{k+1} W''), \rho(\gamma_1(t))[v''_1/x], \rho(\gamma_2(t'))[v''_2/x]) \in \mathcal{T}_\rho[\![G_2]\!]$$

If the following term reduces to error, then the result follows immediately.

$$W''.\Xi_1 \triangleright \rho(\gamma_1(t))[v''_1/x]$$

If the above is not true, then the following terms reduce to values (v_{if}) and $\exists W''' \geq (\downarrow_{k+1} W'')$ such that $(W''', v_{1f}, v_{2f}) \in \mathcal{V}_\rho[\![G_2]\!]$ and $W'''.j + k^* = (\downarrow_{k+1} W'').j$, or what is the same $W'''.j + k^* + k + 1 = (W'').j$.

$$W''.\Xi_1 \triangleright \rho(\gamma_1(t))[v''_1/x] \longrightarrow^{k^*} W'''.\Xi_1 \triangleright v_{1f}$$

$$W''.\Xi_2 \triangleright \rho(\gamma_2(t'))[v''_2/x] \longrightarrow^* W'''.\Xi_2 \triangleright v_{2f}$$

We instantiate the induction hypothesis in the previous result $((W''', v_{1f}, v_{2f}))$ with the type G'_2 and the evidence $cod(\varepsilon^\rho_i) \vdash W'.\Xi_i \vdash G''_2 \sim G'_2$, then we obtain that:

$$(W''', cod(\varepsilon^\rho_1)v_{1f} :: \rho(G'_2), cod(\varepsilon^\rho_2)v_{2f} :: \rho(G'_2)) \in \mathcal{T}_\rho[\![G'_2]\!]$$

Therefore, we get $(\downarrow_k W''', v^*_1, v^*_2) \in \mathcal{V}_\rho[\![G'_2]\!]$. Taking $W^* = (\downarrow_k W''')$, the result follows immediately. Note that $W'''.j + k + k^* + 1 = W''.j$ and therefore $(\downarrow_k W''').j + 1 + 2k + k^* = W''.j$.

For the other cases of G, let's considerer that $u_1 = \lambda x : \rho(G_1).\rho(\gamma_1(t))$, $u_2 = \lambda x : \rho(\rho(G_1).\rho(\gamma_2(t'))$ and $G^* = G_1 \rightarrow G_2$, we have to prove that:

$$(W, \rho_1(\varepsilon)u_1 :: \rho(G), \rho_2(\varepsilon)u_2 :: \rho(G)) \in \mathcal{V}_\rho[\![G]\!]$$

(2) If $G \in$ TypeName then $\varepsilon = \langle H_3, \alpha^{E_4}\rangle$. Notice that as α^{E_4} cannot have free type variables therefore H_3 neither. Then $\varepsilon = \rho_i(\varepsilon)$. As α is sync, then let us call $G'' = W.\Xi_i(\alpha)$. We have to prove that:

$$(W, \langle H_3, \alpha^{E_4}\rangle\, u_1 :: \alpha, \langle H_3, \alpha^{E_4}\rangle\, u_2 :: \alpha) \in \mathcal{V}_\rho[\![\alpha]\!]$$

which, by definition of $\mathcal{V}_\rho[\![\alpha]\!]$, is equivalent to prove that:

$$(\downarrow W, \langle H_3, E_4\rangle\, u_1 :: G'', \langle E_3, E_4\rangle\, u_2 :: G'') \in \mathcal{V}_\rho[\![G'']\!]$$

Then we proceed by case analysis on ε:

- (Case $\varepsilon = \langle H_3, \alpha^{\beta^{E_4}}\rangle$). We know that $\langle H_3, \alpha^{\beta^{E_4}}\rangle \vdash \Xi; \Delta \vdash G^* \sim \alpha$, then by Lemma 6.29, $\langle H_3, \beta^{E_4}\rangle \vdash \Xi; \Delta \vdash G^* \sim G''$. As $\beta^{E_4} \sqsubseteq G''$, then G'' can either be ? or β.
  If $G'' = ?$, then by definition of $\mathcal{V}_\rho[\![?]\!]$, we have to prove that the resulting values belong to $\mathcal{V}_\rho[\![\beta]\!]$. Also as $\langle H_3, \beta^{E_4}\rangle \vdash \Xi; \Delta \vdash G^* \sim ?$, by Lemma 6.27, $\langle H_3, \beta^{E_4}\rangle \vdash \Xi; \Delta \vdash G^* \sim \beta$, and then we proceed just like this case once again (this is process is finite as there are no circular references by construction and it ends up in something different to a type name).
  If $G'' = \beta$ we use an analogous argument as for $G'' = ?$.
- (Case $\varepsilon = \langle H_3, \alpha^{H_4}\rangle$). We have to prove that

$$(\downarrow W, \langle H_3, H_4\rangle\, u_1 :: G'', \langle H_3, H_4\rangle\, u_2 :: G'') \in \mathcal{V}_\rho[\![G'']\!]$$

  By Lemma 6.29, $\langle H_3, H_4\rangle \vdash \Xi; \Delta \vdash G^* \sim G''$. Then if $G'' = ?$, we proceed as the case $G = ?$, with the evidence $\varepsilon = \langle H_3, H_4\rangle$. If $G'' \in$ HeadType, we proceed as the previous case where $G = G_1' \rightarrow G_2'$, and the evidence $\varepsilon = \langle H_3, H_4\rangle$.
  Also, we have to prove that $(\forall \Xi', \varepsilon', G_1^*$, such that $\varepsilon'.n = k$, $\varepsilon' = \langle \alpha^{E_1^{**}}, E_2^{**}\rangle$ $(\downarrow W \in \mathcal{S}[\![\Xi']\!] \wedge \varepsilon' \vdash \Xi' \vdash \alpha \sim G_1^*)$, we get that

$$(\downarrow_1 W, \varepsilon'(\langle H_3, \alpha^{H_4}\rangle u_1 :: \alpha) :: G_1^*, \varepsilon'(\langle H_4, \alpha^{E_{22}}\rangle u_2 :: \alpha) :: G_1^*) \in \mathcal{T}_\rho[\![G_1^*]\!])$$

  or what is the same $((\langle H_3, \alpha^{H_4}\rangle \mathbin{⨾} \varepsilon')$ fails the result follows immediately)

$$(\downarrow_{1+k} W, (\langle H_3, \alpha^{H_4}\rangle \mathbin{⨾} \varepsilon')u_1 :: G_1^*, (\langle H_2, \alpha^{H_4}\rangle \mathbin{⨾} \varepsilon')u_2 :: G_1^*) \in \mathcal{V}_\rho[\![G_1^*]\!])$$

  By definition of transitivity and Lemma 6.30, we know that

$$\langle H_3, \alpha^{H_4}\rangle \mathbin{⨾} \langle \alpha^{E_1^{**}}, E_2^{**}\rangle = \langle H_3, H_4\rangle \mathbin{⨾} \langle E_1^{**}, E_2^{**}\rangle$$

  We know that $\langle E_1^{**}, E_2^{**}\rangle \vdash \Xi' \vdash G'' \sim G_1^*$. Since $\langle E_1^{**}, E_2^{**}\rangle \vdash \Xi \vdash G'' \sim G_1^*$, $\downarrow_1 W \in \mathcal{S}[\![\Xi']\!]$, $(\downarrow_1 W, \langle H_3, H_4\rangle u_1 :: G'', \langle H_1, H_4\rangle u_2 :: G'') \in \mathcal{V}_\rho[\![G'']\!]$, by Lemma 6.17, we know that (since $(\langle H_3, \alpha^{H_4}\rangle \mathbin{⨾} \varepsilon')$ does not fail then $(\langle H_3, H_4\rangle \mathbin{⨾} \langle E_1^{**}, E_2^{**}\rangle)$ also does not fail by the transitivity rules)

$$(\downarrow_{1+k} W, (\langle H_3, H_4\rangle \mathbin{⨾} \langle E_1^{**}, E_2^{**}\rangle)u_1 :: G_1^*, (\langle H_3, H_4\rangle \mathbin{⨾} \langle E_1^{**}, E_2^{**}\rangle)u_2 :: G_1^*) \in \mathcal{V}_\rho[\![G_1^*]\!])$$

  The result follows immediately.

(3) If $G = ?$ we have the following cases:

- $(G = ?, \varepsilon = \langle H_3, H_4\rangle)$. By the definition of $\mathcal{V}_\rho[\![?]\!]$ in this case we have to prove that:

$$(W, \rho_1(\varepsilon)u_1 :: \rho(G), \rho_2(\varepsilon)u_2 :: \rho(G)) \in \mathcal{V}_\rho[\![const(H_4)]\!]$$

but as $const(H_4) = ? \rightarrow ?$, we proceed just like this case where $G = G_1' \rightarrow G_2$, where $G_1' = ?$ and $G_2' = ?$.

- ($G = ?, \varepsilon = \langle H_3, \alpha^{E_4}\rangle$). Notice that as α^{E_4} cannot have free type variables therefore E_3 neither. Then $\varepsilon = \rho_i(\varepsilon)$. By the definition of $\mathcal{V}_\rho[\![?]\!]$ we have to prove that

$$(W, \langle H_3, \alpha^{E_4}\rangle\, u_1 :: \alpha, \langle H_3, \alpha^{E_4}\rangle\, u_2 :: \alpha) \in \mathcal{V}_\rho[\![\alpha]\!]$$

Note that by Lemma 6.27 we know that $\varepsilon \vdash \Xi; \Delta \vdash G^* \sim \alpha$. Then we proceed just like the case $G \in$ TypeName.

□

Lemma 10.2 (Compatibility-EΛ). *If* $\Xi; \Delta, X \vdash t_1 \preceq t_2 : G$, $\varepsilon \vdash \Xi; \Delta \vdash \forall X.G \sim G'$ *and* $\Xi; \Delta \vdash \Gamma$ *then* $\Xi; \Delta; \Gamma \vdash \varepsilon(\Lambda X.t_1) :: G' \preceq \varepsilon(\Lambda X.t_2) :: G' : G'$.

Proof. First, we are required to prove that $\Xi; \Delta; \Gamma \vdash \varepsilon(\Lambda X.t_i) :: G' : G'$, but by unfolding the premises we know that $\Xi; \Delta, X \vdash t_i : G$, therefore:

$$\frac{\Xi; \Delta, X; \Gamma \vdash t_i : G \qquad \Xi; \Delta \vdash \Gamma}{\Xi; \Delta; \Gamma \vdash \Lambda X.t_i \in \forall X.G}$$

Then we can conclude that:

$$\frac{\Xi; \Delta; \Gamma \vdash \Lambda X.t_i \in \forall X.G \qquad \varepsilon \vdash \Xi; \Delta \vdash \forall X.G \sim G'}{\Xi; \Delta; \Gamma \vdash \varepsilon(\Lambda X.t_i) :: G' : G'}$$

Consider arbitrary W, ρ, γ such that $W \in \mathcal{S}[\![\Xi]\!]$, $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$ and $(W, \gamma) \in \mathcal{G}_\rho[\![\Gamma]\!]$. We are required to show that:

$$(W, \rho(\gamma_1(\varepsilon(\Lambda X.t_1) :: G')), \rho(\gamma_2(\varepsilon(\Lambda X.t_2) :: G'))) \in \mathcal{T}_\rho[\![G']\!]$$

First we have to prove that:

$$W.\Xi_i \vdash \rho(\gamma_i(\varepsilon(\Lambda X.t_i) :: G')) : \rho(G')$$

As we know that $\Xi; \Delta; \Gamma \vdash \varepsilon(\Lambda X.t_i) :: G' : G'$, by Lemma 6.25 the result follows immediately.

By definition of substitutions $\rho(\gamma_i(\varepsilon(\Lambda X.t_1) :: G')) = \varepsilon_i^\rho(\Lambda X.\rho(\gamma_i(t_i)))) :: \rho(G')$, where $\varepsilon_i^\rho = \rho_i(\varepsilon)$, therefore we have to prove that:

$$(W, \varepsilon_1^\rho(\Lambda X.\rho(\gamma_1(t_1)))) :: \rho(G'), \varepsilon_2^\rho(\Lambda X.\rho(\gamma_2(t_2)))) :: \rho(G')) \in \mathcal{T}_\rho[\![G']\!]$$

We already know that both terms are values and therefore we only have to prove that:

$$(W, \varepsilon_1^\rho(\Lambda X.\rho(\gamma_1(t_1)))) :: \rho(G'), \varepsilon_2^\rho(\Lambda X.\rho(\gamma_2(t_2)))) :: \rho(G')) \in \mathcal{V}_\rho[\![G']\!]$$

Let's suppose that $\varepsilon_1^\rho.n = k$.

The type G' can be $\forall X.G_1'$, for some G_1', ? or a TypeName. Let $u_1 = \Lambda X.\rho(\gamma_1(t_1))$, $u_2 = \Lambda X.\rho(\gamma_2(t_2))$ and $G^* = \forall X.G$, we have to prove that:

$$(W, \rho_1(\varepsilon)u_1 :: \rho(G), \rho_2(\varepsilon)u_2 :: \rho(G)) \in \mathcal{V}_\rho[\![G']\!]$$

(1) If $G' = \forall X.G_1'$, then consider $W' \geq W$, and G_1, G_2, R and α, such that $W'.\Xi_i \vdash G_i$, and $R \in$ Rel$_{W'.j}[G_1, G_2]$.

$$W'.\Xi_i \triangleright \varepsilon_i^\rho(\Lambda X.\rho(\gamma_i(t_i)))) :: \forall X.\rho(G_1')\ [G_i] \longrightarrow$$

$$W'.\Xi_i, \alpha := G_i \triangleright \varepsilon_{\forall X.\rho(G_1')}^{E_i/\alpha^{E_i}}(\varepsilon_i^\rho[\alpha^{E_i}]\rho(\gamma_i(t_i))[\alpha^{E_i}/X] :: \rho(G_1')[\alpha/X]) :: \rho(G_1')[G_i/X]$$

where $E_i' = lift_{(W'.\Xi_i)}(G_i)$.

Note that $\varepsilon \vdash \Xi; \Delta \vdash \forall X.G \sim \forall X.G_1'$, then $\varepsilon = \langle \forall X.E_1, \forall X.E_2 \rangle$, for some E_1, E_2, K and L. By the Lemma 6.24 we know that $\varepsilon_i^\rho \vdash W.\Xi_i; \Delta \vdash \forall X.\rho(G) \sim \forall X.\rho(G_1')$, then $\varepsilon_i^\rho = \langle \forall X.E_{i1}, \forall X.E_{i2} \rangle$, where $\forall X.E_{i1} = \rho_i(E_1)$ and $E_{i2} = \rho_i(E_2)$.
Then we have to prove that:

$$(W'', (\varepsilon_1^\rho[\alpha^{E_1}])\rho(\gamma_1(t_1))[\alpha^{E_1}/X] :: \rho(G_1')[\alpha/X],$$

$$(\varepsilon_2^\rho[\alpha^{E_2}])\rho(\gamma_2(t_2))[\alpha^{E_2}/X] :: \rho(G_1')[\alpha/X]) \in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G_1']\!]$$

where $W'' = \downarrow(W' \boxtimes (\alpha, G_1, G_2, R))$.
Note that

$$W''.\Xi_1 \triangleright (\varepsilon_1^\rho\ [\![\alpha^{E_1}]\!])\rho(\gamma_1(t_1)) :: \rho(G_1')[\alpha/X] \longmapsto^{k^*}$$

$$\Xi_1 \triangleright (\varepsilon_1^\rho\ [\![\alpha^{E_1}]\!])v_{1f} \longmapsto^{k}$$

$$\Xi_1 \triangleright v_1^*$$

Let $\rho' = \rho[X \mapsto \alpha]$. We instantiate the premise $\Xi; \Delta; \Gamma \vdash t_1 \preceq t_2 : G$ with W'', ρ' and γ, such that $W'' \in \mathcal{S}[\![\Xi]\!]$, as $\alpha \in dom(W'.\kappa[\alpha \mapsto R])$ then $(W'', \rho') \in \mathcal{D}[\![\Delta, X]\!]$. Also note that as X is fresh, then $\forall (v_1^*, v_2^*) \in cod(\gamma)$, such that $\Xi; \Delta; \Gamma \vdash v_i^* : G^*$, $X \notin FV(G^*)$, then it is easy to see that $(W'', \gamma) \in \mathcal{G}_{\rho[X\mapsto\alpha]}[\![\Gamma]\!]$. Then we know that:

$$(W'', \rho'(\gamma_1(t_1)), \rho'(\gamma_2(t_2))) \in \mathcal{T}_{\rho'}[\![G]\!]$$

But note that:

$$\rho'(\gamma_i(t_i)) = \rho[\alpha/X](\gamma_i(t_i)) = \rho(\gamma_i(t_i))[\alpha^{E_i}/X]$$

Then we have that:

$$(W'', \rho(\gamma_1(t_1))[\alpha^{E_1}/X], \rho(\gamma_2(t_2))[\alpha^{E_2}/X]) \in \mathcal{T}_{\rho[\alpha/X]}[\![G]\!]$$

If the following term reduces to error, then the result follows immediately.

$$W''.\Xi_1 \triangleright \rho(\gamma_1(t_1))[\alpha^{E_1}/X]$$

If the above is not true, then the following terms reduce to values ($v_{if} = \varepsilon_{if} u_{if} :: \rho'(G)$) and $\exists W''' \geq W''$ such that $(W''', v_{1f}, v_{2f}) \in \mathcal{V}_{\rho[\alpha\mapsto X]}[\![G]\!]$ and $W'''.j + k^* = W''.j$.

$$W''.\Xi_i \triangleright \rho(\gamma_i(t_i))[\alpha^{E_i}/X] \longrightarrow^* W'''.\Xi_i \triangleright v_{if}$$

We instantiate the Lemma 6.17 with the type G_1' and the evidence $\langle E_1, E_2 \rangle \vdash \Xi; \Delta, X \vdash G \sim G_1'$ (remember that $\varepsilon = \langle \forall X.E_1, \forall X.E_2 \rangle$). Note that $\varepsilon_i^\rho\ [\![\alpha^{E_i}]\!] = \rho[X \mapsto \alpha]_{W'''.\Xi_i}(\langle E_1, E_2 \rangle)$, $\rho[X \mapsto \alpha](G_1') = \rho(G_1')[\alpha/X]$, $W''' \in \mathcal{S}[\![\Xi]\!]$ and $(W''', \rho[X \mapsto \alpha]) \in \mathcal{D}[\![\Delta, X]\!]$. Then we obtain that:

$$(W''', (\varepsilon_1^\rho\ [\![\alpha^{E_1}]\!])v_{1f} :: \rho(G_1')[\alpha/X], (\varepsilon_2^\rho\ [\![\alpha^{E_2}]\!])v_{2f} :: \rho(G_1')[\alpha/X]) \in \mathcal{T}_{\rho'}[\![G_1']\!]$$

and

$$(\downarrow_k W''', v_1^*, v_2^*) \in \mathcal{T}_\rho[\![G_1']\!]$$

where $(\downarrow_k W''').j + k + k^* = W''.j$ and $v_i^* = (\varepsilon_{if} \mathbin{;} (\varepsilon_1^\rho\ [\![\alpha^{E_1}]\!]))u_{if} :: \rho(G_1')[\alpha/X]$, and the result follows immediately.

(2) If $G' \in$ TypeName then $\varepsilon = \langle H_3, \alpha^{E_4} \rangle$. Notice that as α^{E_4} cannot have free type variables therefore H_3 neither. Then $\varepsilon = \rho_i(\varepsilon)$. As α is sync, then let us call $G'' = W.\Xi_i(\alpha)$. We have to prove that:

$$(W, \langle H_3, \alpha^{E_4} \rangle\, u_1 :: \alpha, \langle H_3, \alpha^{E_4} \rangle\, u_2 :: \alpha) \in \mathcal{V}_\rho[\![\alpha]\!]$$

which, by definition of $\mathcal{V}_\rho[\![\alpha]\!]$, is equivalent to prove that:

$$(\downarrow W, \langle H_3, E_4 \rangle\, u_1 :: G'', \langle E_3, E_4 \rangle\, u_2 :: G'') \in \mathcal{V}_\rho[\![G'']\!]$$

Then we proceed by case analysis on ε:

- (Case $\varepsilon = \langle H_3, \alpha^{\beta^{E_4}} \rangle$). We know that $\langle H_3, \alpha^{\beta^{E_4}} \rangle \vdash \Xi; \Delta \vdash G^* \sim \alpha$, then by Lemma 6.29, $\langle H_3, \beta^{E_4} \rangle \vdash \Xi; \Delta \vdash G^* \sim G''$. As $\beta^{E_4} \sqsubseteq G''$, then G'' can either be ? or β.
  If $G'' = ?$, then by definition of $\mathcal{V}_\rho[\![?]\!]$, we have to prove that the resulting values belong to $\mathcal{V}_\rho[\![\beta]\!]$. Also as $\langle H_3, \beta^{E_4} \rangle \vdash \Xi; \Delta \vdash G^* \sim ?$, by Lemma 6.27, $\langle H_3, \beta^{E_4} \rangle \vdash \Xi; \Delta \vdash G^* \sim \beta$, and then we proceed just like this case once again (this is process is finite as there are no circular references by construction and it ends up in something different to a type name).
  If $G'' = \beta$ we use an analogous argument as for $G'' = ?$.
- (Case $\varepsilon = \langle H_3, \alpha^{H_4} \rangle$). We have to prove that

$$(\downarrow W, \langle H_3, H_4 \rangle\, u_1 :: G'', \langle H_3, H_4 \rangle\, u_2 :: G'') \in \mathcal{V}_\rho[\![G'']\!]$$

  By Lemma 6.29, $\langle H_3, H_4 \rangle \vdash \Xi; \Delta \vdash G^* \sim G''$. Then if $G'' = ?$, we proceed as the case $G' = ?$, with the evidence $\varepsilon = \langle H_3, H_4 \rangle$. If $G'' \in$ HeadType, we proceed as the previous case where $G' = \forall X.G$, and the evidence $\varepsilon = \langle H_3, H_4 \rangle$.
  Also, we have to prove that $(\forall \Xi', \varepsilon', G_1^*$, such that $\varepsilon'.n = k$, $\varepsilon' = \langle \alpha^{E_1^{**}}, E_2^{**} \rangle$ $(\downarrow W \in \mathcal{S}[\![\Xi']\!] \wedge \varepsilon' \vdash \Xi' \vdash \alpha \sim G_1^*)$, we get that

$$(\downarrow_1 W, \varepsilon'(\langle H_3, \alpha^{H_4} \rangle u_1 :: \alpha) :: G_1^*, \varepsilon'(\langle H_4, \alpha^{E_{22}} \rangle u_2 :: \alpha) :: G_1^*) \in \mathcal{T}_\rho[\![G_1^*]\!])$$

  or what is the same (($\langle H_3, \alpha^{H_4} \rangle \mathbin{⨟} \varepsilon'$) fails the result follows immediately)

$$(\downarrow_{1+k} W, (\langle H_3, \alpha^{H_4} \rangle \mathbin{⨟} \varepsilon')u_1 :: G_1^*, (\langle H_2, \alpha^{H_4} \rangle \mathbin{⨟} \varepsilon')u_2 :: G_1^*) \in \mathcal{V}_\rho[\![G_1^*]\!])$$

  By definition of transitivity and Lemma 6.30, we know that

$$\langle H_3, \alpha^{H_4} \rangle \mathbin{⨟} \langle \alpha^{E_1^{**}}, E_2^{**} \rangle = \langle H_3, H_4 \rangle \mathbin{⨟} \langle E_1^{**}, E_2^{**} \rangle$$

  We know that $\langle E_1^{**}, E_2^{**} \rangle \vdash \Xi' \vdash G'' \sim G_1^*$. Since $\langle E_1^{**}, E_2^{**} \rangle \vdash \Xi \vdash G'' \sim G_1^*$, $\downarrow_1 W \in \mathcal{S}[\![\Xi']\!]$, $(\downarrow_1 W, \langle H_3, H_4 \rangle u_1 :: G'', \langle H_1, H_4 \rangle u_2 :: G'') \in \mathcal{V}_\rho[\![G'']\!]$, by Lemma 6.17, we know that (since $(\langle H_3, \alpha^{H_4} \rangle \mathbin{⨟} \varepsilon')$ does not fail then $(\langle H_3, H_4 \rangle \mathbin{⨟} \langle E_1^{**}, E_2^{**} \rangle)$ also does not fail by the transitivity rules)

$$(\downarrow_{1+k} W, (\langle H_3, H_4 \rangle \mathbin{⨟} \langle E_1^{**}, E_2^{**} \rangle)u_1 :: G_1^*, (\langle H_3, H_4 \rangle \mathbin{⨟} \langle E_1^{**}, E_2^{**} \rangle)u_2 :: G_1^*) \in \mathcal{V}_\rho[\![G_1^*]\!])$$

  The result follows immediately.

(3) If $G' = ?$ we have the following cases:

- ($G' = ?, \varepsilon = \langle H_3, H_4 \rangle$). By the definition of $\mathcal{V}_\rho[\![?]\!]$ in this case we have to prove that:

$$(W, \rho_1(\varepsilon)u_1 :: \rho(G), \rho_2(\varepsilon)u_2 :: \rho(G)) \in \mathcal{V}_\rho[\![const(H_4)]\!]$$

  but as $const(H_4) = \forall X.?$, we proceed just like the case where $G' = \forall X.G_1'$, where $G_1' = ?$.
- ($G' = ?, \varepsilon = \langle H_3, \alpha^{E_4} \rangle$). Notice that as α^{E_4} cannot have free type variables therefore E_3 neither. Then $\varepsilon = \rho_i(\varepsilon)$. By the definition of $\mathcal{V}_\rho[\![?]\!]$ we have to prove that

$$(W, \langle H_3, \alpha^{E_4} \rangle\, u_1 :: \alpha, \langle H_3, \alpha^{E_4} \rangle\, u_2 :: \alpha) \in \mathcal{V}_\rho[\![\alpha]\!]$$

Note that by Lemma 6.27 we know that $\varepsilon \vdash \Xi; \Delta \vdash G^* \sim \alpha$. Then we proceed just like the case $G' \in$ TypeName.

□

Proposition 6.5 (**Compatibility-EpairU**). *If* $\Xi; \Delta; \Gamma \vdash \pi_1(\varepsilon)u_1 :: G_1 \leq \pi_1(\varepsilon)u_1' :: G_1 : G_1$, $\Xi; \Delta; \Gamma \vdash \pi_2(\varepsilon)u_2' :: G_2 \leq \pi_2(\varepsilon)u_2' :: G_2 : G_2$, *and* $\varepsilon \Vdash \Xi; \Delta \vdash G_1 \times G_2 \sim G$ *then:*

$$\Xi; \Delta; \Gamma \vdash \varepsilon\langle u_1, u_2\rangle :: G \leq \varepsilon\langle u_1', u_2'\rangle :: G : G$$

Proof. Straightforward as the definition of related pairs depends on a weaker property of the premise: $\Xi; \Delta; \Gamma \vdash \pi_1(\varepsilon)u_1 :: G_1 \leq \pi_1(\varepsilon)u_1' :: G_1 : G_1$ and $\Xi; \Delta; \Gamma \vdash \pi_2(\varepsilon)u_2' :: G_2 \leq \pi_2(\varepsilon)u_2' :: G_2 : G_2$. □

Proposition 6.6 (**Compatibility-Epair**). *If* $\Xi; \Delta; \Gamma \vdash t_1 \leq t_1' : G_1$ *and* $\Xi; \Delta; \Gamma \vdash t_2 \leq t_2' : G_2$, *then* $\Xi; \Delta; \Gamma \vdash \langle t_1, t_2\rangle \leq \langle t_1', t_2'\rangle : G_1 \times G_2$.

Proof. We proceed by induction on subterms t_i, analogous to the function application case, but using Prop 6.5 instead. □

Proposition 6.7 (**Compatibility-Ex**). *If* $x : G \in \Gamma$ *and* $\Xi; \Delta \vdash \Gamma$ *then* $\Xi; \Delta; \Gamma \vdash x \leq x : G$.

Proof. First, we are required to show $\Xi; \Delta; \Gamma \vdash x : G$, which is immediate. Consider arbitrary W, ρ, γ such that $W \in \mathcal{S}[\![\Xi]\!]$, $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$ and $(W, \gamma) \in \mathcal{G}_\rho[\![\Gamma]\!]$. We are required to show that:

$$(W, \rho(\gamma_1(x)), \rho(\gamma_2(x))) \in \mathcal{T}_\rho[\![G]\!]$$

Consider arbitrary i, v_1 and Ξ_1 such that $i < W.j$ and $W.\Xi_1 \triangleright \rho(\gamma_1(x)) \longrightarrow^i \Xi_1 \triangleright v_1$. Since $\rho(\gamma_1(x))) = \gamma_1(x)$ and $\gamma_1(x)$ is already a value, we have $i = 0$ and $\gamma_1(x) = v_1$. We are required to show that exists Ξ_2, v_2 such that $W.\Xi_2 \triangleright \gamma_2(x) \longrightarrow^* \Xi_2 \triangleright v_2$ which is immediate (since $\rho(\gamma_2(x)) = \gamma_2(x)$ is a value and $\Xi_2 = W.\Xi_2$). Also, we are required to show that $\exists W'$, such that $W'.j + i = W.j \wedge W' \geq W \wedge W'.\Xi_1 = \Xi_1 \wedge W'.\Xi_2 = \Xi_2 \wedge (W', \gamma_1(x), \gamma_2(x)) \in \mathcal{V}_\rho[\![G]\!]$. Let $W' = W$, then $(W, \gamma_1(x), \gamma_2(x)) \in \mathcal{V}_\rho[\![G]\!]$ because of the definition of $(W, \gamma) \in \mathcal{G}_\rho[\![\Gamma]\!]$. □

Proposition 6.8 (**Compatibility-Easc**). *If* $\Xi; \Delta; \Gamma \vdash t_1 \leq t_2 : G$ *and* $\varepsilon \vdash \Xi; \Delta \vdash G \sim G'$ *then* $\Xi; \Delta; \Gamma \vdash \varepsilon t_1 :: G' \leq \varepsilon t_2 :: G' : G'$.

Proof. First we are required to prove that $\Xi; \Delta; \Gamma \vdash \varepsilon t_i :: G' : G'$, but by $\Xi; \Delta; \Gamma \vdash t_1 \leq t_2 : G$ we already know that $\Xi; \Delta; \Gamma \vdash t_i : G$, therefore:

$$(\text{Easc})\frac{\Xi; \Delta; \Gamma \vdash t_i : G \qquad \varepsilon \vdash \Xi; \Delta \vdash G \sim G'}{\Xi; \Delta; \Gamma \vdash \varepsilon t_i :: G' : G'}$$

Consider arbitrary W, ρ, γ such that $W \in \mathcal{S}[\![\Xi]\!]$, $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$ and $(W, \gamma) \in \mathcal{G}_\rho[\![\Gamma]\!]$. We are required to show that:

$$(W, \rho(\gamma_1(\varepsilon t_1 :: G')), \rho(\gamma_2(\varepsilon t_2 :: G'))) \in \mathcal{T}_\rho[\![G']\!]$$

Let's suppose that $\varepsilon_1^\rho.n = k$. By definition of substitutions $\rho(\gamma_i(\varepsilon t_i :: G')) = \rho(\varepsilon)\rho(\gamma_i(t_i)) :: \rho(G')$, therefore we have to prove that:

$$(W, \rho(\varepsilon)\rho(\gamma_1(t_1)) :: \rho(G'), \rho(\varepsilon)\rho(\gamma_2(t_2)) :: \rho(G')) \in \mathcal{T}_\rho[\![G']\!]$$

First we have to prove that:

$$W.\Xi_i \vdash \rho(\varepsilon)\rho(\gamma_i(t_i)) :: \rho(G') : G'$$

As we know that $\Xi; \Delta; \Gamma \vdash \varepsilon t_i :: G' : G'$, by Lemma 6.25 the result follows immediately.

Second, consider arbitrary $i < W.j, \Xi_1$. Either there exist v_1 such that:

$$W.\Xi_1 \triangleright \rho(\varepsilon)\rho(\gamma_1(t_1)) :: \rho(G') \longmapsto^i \Xi_1 \triangleright v_1$$

or

$$W.\Xi_1 \triangleright \rho(\varepsilon)\rho(\gamma_1(t_1)) :: \rho(G') \longmapsto^i \mathbf{error}$$

Let us suppose that $W.\Xi_1 \triangleright \rho(\gamma_1(t_1)) :: \rho(G') \longmapsto^i \Xi_1 \triangleright v_1$. Hence, by inspection of the operational semantics, it follows that there exist $i_1 + 1 < i$, Ξ_{11} and v_{11} such that:

$$W.\Xi_1 \triangleright \rho(\varepsilon)\rho(\gamma_1(t_1)) :: \rho(G') \longmapsto^{i_1} \Xi_{11} \triangleright \rho(\varepsilon)v_{11} :: \rho(G') \longmapsto^k \Xi_{11} \triangleright v_1$$

We instantiate the hypothesis $\Xi; \Delta; \Gamma \vdash t_1 \preceq t_2 : G$ with W, ρ and γ to obtain that:

$$(W, \rho(\gamma_1(t_1)), \rho(\gamma_2(t_2))) \in \mathcal{T}_\rho[\![G]\!]$$

We instantiate $\mathcal{T}_\rho[\![G]\!]$ with i_1, Ξ_{11} and v_{11} (note that $i_1 < i < W.j$), hence there exists v_{12} and W_1, such that $W_1 \geq W$, $W_1.j + i_1 = W.j$, $W.\Xi_2 \triangleright \rho(\gamma_2(t_2)) \longmapsto^* W_1.\Xi_2 \triangleright v_{12}$, $W_1.\Xi_1 = \Xi_{11}$, v_{12} and $(W_1, v_{11}, v_{12}) \in \mathcal{V}_\rho[\![G]\!]$.

Since we have that $(W_1, v_{11}, v_{12}) \in \mathcal{V}_\rho[\![G]\!]$, then it is true that $(W_1, \rho(\varepsilon)v_{11} :: G', \rho(\varepsilon)v_{12} :: G') \in \mathcal{T}_\rho[\![G']\!]$ by the Lemma 6.17.

By the inspection of the operational semantics:

$$W.\Xi_1 \triangleright \rho(\varepsilon)\rho(\gamma_1(t_1)) :: \rho(G') \longmapsto^{i_1} W_1.\Xi_1 \triangleright \rho(\varepsilon)v_{11} :: \rho(G') \longmapsto^k \Xi_1 \triangleright v_1$$

We instantiate $(W_1, \rho(\varepsilon)v_{11} :: G', \rho(\varepsilon)v_{12} :: G') \in \mathcal{T}_\rho[\![G']\!]$ with k, v_1 and Ξ_1. Therefore there must exist v_2 and W' such that $W' \geq W_1$ (note that $W' \geq W$), $W'.j + i_1 + k = W'.j + i = W.j$.

$$W_1.\Xi_2 \triangleright \rho(\varepsilon)v_{12} :: \rho(G') \longmapsto^* \Xi_2 \triangleright v_2$$

and $(W', v_1, v_2) \in \mathcal{V}_\rho[\![G']\!]$ then the result follows. □

Proposition 6.9 (**Compatibility-Eop**). *If* $\Xi; \Delta; \Gamma \vdash \overline{t} \preceq \overline{t'} : \overline{G}$ *and* $ty(op) = \overline{G} \to G$ *then* $\Xi; \Delta; \Gamma \vdash op(\overline{t}) \preceq op(\overline{t'}) : G$.

Proof. Similar to the term application. □

Proposition 6.10 (**Compatibility-Eapp**). *If* $\Xi; \Delta; \Gamma \vdash t_1 \preceq t_1' : G_{11} \to G_{12}$ *and* $\Xi; \Delta; \Gamma \vdash t_2 \preceq t_2' : G_{11}$ *then* $\Xi; \Delta; \Gamma \vdash t_1\ t_2 \preceq t_1'\ t_2' : G_{12}$.

Proof. First, we are required to show that:

$$\Xi; \Delta; \Gamma \vdash t_1\ t_2 : G_{12}$$

which follows directly from (Eapp) as $\Xi; \Delta; \Gamma \vdash t_1 : G_1$, and $\Xi; \Delta; \Gamma \vdash t_2 : G_2$. Also, we are required to prove that:

$$\Xi; \Delta; \Gamma \vdash t_1'\ t_2' : G_{12}$$

which follows analogously.

Second, consider arbitrary W, ρ, γ such that $W \in \mathcal{S}[\![\Xi]\!]$, $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$ and $(W, \gamma) \in \mathcal{G}_\rho[\![\Gamma]\!]$. We are required to show that:

$$(W, \rho(\gamma_1(t_1\ t_2)), \rho(\gamma_2(t_1'\ t_2')) \in \mathcal{T}_\rho[\![G_{12}]\!]$$

Consider arbitrary i, v_1 and Ξ_1 such that $i < W.j$ and:

$$W.\Xi_1 \triangleright \rho(\gamma_1(t_1\ t_2)) \longrightarrow^i \Xi_1 \triangleright v_1 \vee W.\Xi_1 \triangleright \rho(\gamma_1(t_1\ t_2)) \longrightarrow^i \mathbf{error}$$

Hence, by inspection of the operational semantics, it follows that there exist $i_1 < i$, Ξ_{11} and v_{11} such that:

$$W.\Xi_1 \triangleright \rho(\gamma_1(t_1)) \longrightarrow^{i_1} \Xi_{11} \triangleright v_{11} \vee W.\Xi_1 \triangleright \rho(\gamma_1(t_1)) \longrightarrow^{i_1} \textbf{error}$$

If $W.\Xi_1 \triangleright \rho(\gamma_1(t_1)) \longrightarrow^{i_1}$ **error** then $W.\Xi_1 \triangleright \rho(\gamma_2(t_1')) \longrightarrow^{*}$ **error** and the result holds immediately. Let us assume that the reduction does not fail. We instantiate the hypothesis $\Xi; \Delta; \Gamma \vdash t_1 \preceq t_1' : G_{11} \to G_{12}$ with W, ρ and γ we obtain that:

$$(W, \rho(\gamma_1(t_1))), \rho(\gamma_2(t_1'))) \in \mathcal{T}_\rho[\![G_{11} \to G_{12}]\!]$$

We instantiate this with i_1, Ξ_{11} and v_{11} (note that $i_1 < i < W.j$), hence there exists v_{11}' and W_1, such that $W_1 \geq W$, $W_1.j + i_1 = W.j$, or what is the same $W_1.j + i_1 = W.j$, $W.\Xi_2 \triangleright \rho(\gamma_2(t_1')) \longrightarrow^{*} W_1.\Xi_2 \triangleright v_{11}'$, $W_1.\Xi_1 = \Xi_{11}$ and $(W_1, v_{11}, v_{11}') \in \mathcal{V}_\rho[\![G_{11} \to G_{12}]\!]$.

Note that:

$$W.\Xi_1 \triangleright \rho(\gamma_1(t_1\ t_2)) \longrightarrow^{i_1} \Xi_{11} \triangleright v_{11}(\rho(\gamma_1(t_2))) \longrightarrow^{i-i_1} \Xi_1 \triangleright v_1$$

or

$$W.\Xi_1 \triangleright \rho(\gamma_1(t_1\ t_2)) \longrightarrow^{i_1} \Xi_{11} \triangleright v_{11}(\rho(\gamma_1(t_2))) \longrightarrow^{i-i_1} \textbf{error}$$

Hence, by inspection of the operational semantics, it follows that there exist $i_2 < i - i_1$, Ξ_{22} and v_{22} such that:

$$\Xi_{11} \triangleright \rho(\gamma_1(t_2)) \longrightarrow^{i_2} \Xi_{22} \triangleright v_{22} \vee \Xi_{11} \triangleright \rho(\gamma_1(t_2)) \longrightarrow^{i_2} \textbf{error}$$

We instantiate the hypothesis $\Xi; \Delta; \Gamma \vdash t_2 \preceq t_2' : G_{11}$ with (W_1), ρ and γ, then we obtain that:

$$(W_1, \rho(\gamma_1(t_2)), \rho(\gamma_2(t_2'))) \in \mathcal{T}_\rho[\![G_{11}]\!]$$

If $\Xi_{11} \triangleright \rho(\gamma_1(t_2)) \longrightarrow^{i_2}$ **error** then we instantiate with Ξ_{22} and $\Xi_{22} \triangleright \rho(\gamma_2(t_2')) \longrightarrow^{*}$ **error** and the result holds immediately. Let us assume that the reduction does not fail. We instantiate this with i_2 (note that $i_2 < i - i_1 < W_1.j = W.j - i_1$), Ξ_{22} and v_{22}, hence there exists v_{22}' and W_2, such that $W_2.\Xi_1 = \Xi_{22}$, $W_2 \geq W_1$, or what is the same, $W_2 \geq W_1$, $W_2.j = W_1.j - i_2$ ($W_2.j + i_2 + i_1 = W.j$) and

$$W_1.\Xi_2 \triangleright \rho(\gamma_2(t_2')) \longrightarrow^{*} W_2.\Xi_2 \triangleright v_{22}'$$

and $(W_2, v_{22}, v_{22}') \in \mathcal{V}_\rho[\![G_{11}]\!]$.

Note that:

$$W.\Xi_1 \triangleright \rho(\gamma_1(t_1\ t_2)) \longrightarrow^{i_1} \Xi_{11} \triangleright v_{11}\ (\rho(\gamma_1(t_2))) \longrightarrow^{i_2} \Xi_{22} \triangleright v_{11}\ v_{22} \longrightarrow^{i-i_1-i_2} \Xi_1 \triangleright v_1$$

Since $(W_1, v_{11}, v_{11}') \in \mathcal{V}_\rho[\![G_{11} \to G_{12}]\!]$, we instantiate this with W_2, $\rho(G_{11} \to G_{12})$, v_{22} and v_{22}' (note that $(W_2, v_{22}, v_{22}') \in \mathcal{V}_\rho[\![G_{11}]\!]$, $(\downarrow_1 W_2, v_{22}, v_{22}') \in \mathcal{V}_\rho[\![G_{11}]\!]$ and $W_2 \geq W_1$). Then $(W_2, v_{11}\ v_{22}, v_{11}'\ v_{22}') \in \mathcal{T}_\rho[\![G_2]\!]$.

Since $(W_2, v_{11}\ v_{22}, v_{11}'\ v_{22}') \in \mathcal{T}_\rho[\![G_2]\!]$, we instantiate this with $i - i_1 - i_2$ (note that $i - i_1 - i_2 < W_2.j = W.j - i_1 - i_2$ since $i < W.j$), v_1 and Ξ_1.

If $W_2.\Xi_1 \triangleright v_{11}\ v_{22} \longrightarrow^{i-i_1-i_2}$ **error** then $W_2.\Xi_2 \triangleright v_{11}'\ v_{22}' \longrightarrow^{*}$ **error** and the result holds. Let us assume that the reduction does not fail. Hence there exists v_2 and W', such that $W' \geq W_2$ (note that $W' \geq W$), $W'.j = W_2.j - (i - i_1 - i_2) = W.j - i$, $W_2.\Xi_2 \triangleright v_{11}'\ v_{22}' \longrightarrow^{*} W'.\Xi_2 \triangleright v_2$, $W'.\Xi_1 = \Xi_1$ and $(W', v_1, v_2) \in \mathcal{V}_\rho[\![G_{12}]\!]$, then the proof is complete. ◻

Lemma 10.3 (**Compatibility-EappG**). *If* $\Xi; \Delta; \Gamma \vdash t_1 \preceq t_2 : \forall X.G$ *and* $\Xi; \Delta \vdash G'$, *then* $\Xi; \Delta; \Gamma \vdash t_1[G'] \preceq t_2[G'] : G[G'/X]$.

Proof. First we are required to prove that $\Xi;\Delta;\Gamma \vdash t_i[G'] : G[G'/X]$, but by $\Xi;\Delta;\Gamma \vdash t_1 \preceq t_2 : \forall X.G$ we already know that $\Xi;\Delta;\Gamma \vdash t_i : \forall X.G$, therefore:

$$\text{(EappG)}\frac{\Xi;\Delta;\Gamma \vdash t_i : \forall X.G \qquad \Xi;\Delta \vdash G'}{\Xi;\Delta;\Gamma \vdash t_i[G'] : G[G'/X]}$$

Consider arbitrary W, ρ, γ such that $W \in \mathcal{S}[\![\Xi]\!]$, $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$ and $(W, \gamma) \in \mathcal{G}_\rho[\![\Gamma]\!]$. We are required to show that:

$$(W, \rho(\gamma_1(t_1[G'])), \rho(\gamma_2(t_2[G']))) \in \mathcal{T}_\rho[\![G[G'/X]]\!]$$

But by definition of substitutions $\rho(\gamma_i(t_i[G'])) = \rho(\gamma_i(t_i))[\rho(G')]$, therefore we have to prove that:

$$(W, \rho(\gamma_1(t_1))[\rho(G')], \rho(\gamma_2(t_2))[\rho(G')]) \in \mathcal{T}_\rho[\![G[G'/X]]\!]$$

First we have to prove that:

$$W.\Xi_i \vdash \rho(\gamma_i(t_i))[\rho(G')] : \rho(G)[\rho(G')/X]$$

As we know that $\Xi;\Delta;\Gamma \vdash t_i[G'] : G[G'/X]$, by Lemma 6.25 the result follows immediately. Second, consider arbitrary $i < W.j$ and Ξ_1. Either there exist v_1 such that $W.\Xi_1 \triangleright \rho(\gamma_1(t_1))[\rho(G')] \longmapsto^i \Xi_1 \triangleright v_1$ or $W.\Xi_1 \triangleright \rho(\gamma_1(t_1))[\rho(G')] \longmapsto^i \Xi_1 \triangleright \mathbf{error}$. First, let us suppose that:

$$W.\Xi_1 \triangleright \rho(\gamma_1(t_1))[\rho(G')] \longmapsto^i \Xi_1 \triangleright v_1$$

Hence, by inspection of the operational semantics, it follows that there exist $i_1 < i$, and v_{11} such that

$$W.\Xi_1 \triangleright \rho(\gamma_1(t_1))[\rho(G')] \longmapsto^{i_1} \Xi_{11} \triangleright v_{11}[\rho(G')]$$

We instantiate the premise $\Xi;\Delta;\Gamma \vdash t_1 \preceq t_2 : \forall X.G$ with W, ρ and γ to obtain that:

$$(W, \rho(\gamma_1(t_1)), \rho(\gamma_2(t_2))) \in \mathcal{T}_\rho[\![\forall X.G]\!]$$

We instantiate $\mathcal{T}_\rho[\![\forall X.G]\!]$ with i_1, Ξ_{11} and v_{11} (note that $i_1 < i < W.j$), hence there exists v_{12} and W_1, such that $W_1 \succeq W$, $W_1.j = W.j - i_1$, $W.\Xi_2 \triangleright \rho(\gamma_2(t_2)) \longmapsto^* W_1.\Xi_2 \triangleright v_{12}$, $W_1.\Xi_1 = \Xi_{11}$, v_{12} and:

$$(W_1, v_{11}, v_{12}) \in \mathcal{V}_\rho[\![\forall X.G]\!]$$

Then by inspection of the operational semantics:

$$\begin{aligned} W.\Xi_i \triangleright \rho(\gamma_i(t_i))[\rho(G')] &\longmapsto^* W_1.\Xi_i \triangleright v_{1i}[\rho(G')] \\ &\longmapsto W_1.\Xi_i, \alpha := \rho(G') \triangleright \varepsilon_i(\varepsilon_i' t_i' :: \rho(G)[\alpha/X]) :: \rho(G)[\rho(G')/X] \end{aligned}$$

for some ε_1, ε_2, ε_1', ε_2', t_i' and $\alpha \notin dom(W_1.\Xi_i)$. Let us call $t_i'' = (\varepsilon_i' t_i' :: \rho(G)[\alpha/X])$. We instantiate $\mathcal{V}_\rho[\![\forall X.G]\!]$ with α, t_i'', $\rho(G')$, $R = \mathcal{V}_\rho[\![G']\!]$, ε_1, ε_2 and W_1.

Then $(W_1', t_1'', t_2'') \in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G]\!]$, where $W_1' = (\downarrow W_1) \boxtimes (\alpha, \rho(G'), \rho(G'), \mathcal{V}_\rho[\![G']\!])$.

We instantiate $\mathcal{T}_{\rho[X\mapsto\alpha]}[\![G]\!]$ with i_2, Ξ_1, v_1', such that

$$W_1.\Xi_1 \triangleright (\varepsilon_1' t_1' :: \rho(G)[\alpha/X]) \longmapsto^{i_2} \Xi_1 \triangleright v_1'$$

Note that $i_2 < W_1'.j = W.j - i_1 - 1$, since $i < W.j$. Therefore there must exist v_2', and W' such that $W' \succeq W_1'$ (note that $W' \succeq W$), $W'.j + i_1 + 1 + i_2 = W.j - i$,

$$W_1.\Xi_2 \triangleright (\varepsilon_2' t_2' :: \rho(G)[\alpha/X]) \longmapsto^* W'.\Xi_2 \triangleright (\varepsilon_2' v_2'' :: \rho(G)[\alpha/X]) \longmapsto W'.\Xi_2 \triangleright v_2'$$

$W'.\Xi_1 = \Xi_1$ and $(W', v_1', v_2') \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![G]\!]$.

Notice that t_i reduce to a type abstraction of the form $v_{1i} = \langle \forall X.E_{i1}, \forall X.E_{i2} \rangle \Lambda X.t_i''' :: \forall X.\rho(G)$. Let us call $v_i' = \varepsilon_i''' u_i''' :: \rho(G)[\alpha/X])$, as $\pi_2(\varepsilon_1''') \equiv \pi_2(\varepsilon_2''')$, then $G_p = unlift(\pi_2(\varepsilon_i'''))$, then $E_i =$

$lift_{W_2.\Xi_i}(G_p)$, and $E'_i = lift_{W_1.\Xi_i}(\rho(G'))$, and $\varepsilon_i = \langle E_i[\alpha^{E'_i}/X], E_i[E'_i/X]\rangle$. Then as $(W', v'_1, v'_2) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![G]\!]$ by Lemma 6.15,

$$(\downarrow_k W', (\varepsilon'''_1 \mathbin{\text{;}} \varepsilon_1)u'''_1 :: \rho(G)[\rho(G')/X], (\varepsilon'''_2 \mathbin{\text{;}} \varepsilon_2)u'''_2 :: \rho(G)[\rho(G')/X]) \in \mathcal{V}_\rho[\![G[G'/X]]\!]$$

where $\varepsilon_1.n = k$. Let us call $v_i = (\varepsilon'''_i \mathbin{\text{;}} \varepsilon_i)u'''_1 :: \rho(G)[\rho(G')/X]$. Where the lemma holds by instantiating $\mathcal{T}_\rho[\![G[G'/X]]\!]$ with Ξ_1, v_1, $i = k$ and therefore $W'.\Xi_1 \triangleright \varepsilon_1 v'_1 :: \rho(G)[\rho(G')/X] \longmapsto^k W'.\Xi_1 \triangleright v_1$. Then there must exists some v_2 such that $W'.\Xi_2 \triangleright \varepsilon_2 v'_2 :: \rho(G)[\rho(G')/X] \longmapsto W'.\Xi_2 \triangleright v_2$, and the result follows.

□

Proposition 6.11 (**Compatibility-Epair1**). *If* $\Xi;\Delta;\Gamma \vdash t_1 \preceq t_2 : G_1 \times G_2$ *then* $\Xi;\Delta;\Gamma \vdash \pi_1(t_1) \preceq \pi_1(t_2) : G_1$.

Proof. Similar to the function application case, using the definition of related pairs instead. □

Proposition 6.12 (**Compatibility-Epair2**). *If* $\Xi;\Delta;\Gamma \vdash t_1 \preceq t_2 : G_1 \times G_2$ *then* $\Xi;\Delta;\Gamma \vdash \pi_2(t_1) \preceq \pi_2(t_2) : G_2$.

Proof. Similar to the function application case, using the definition of related pairs instead. □

Lemma 6.13. *Let* $E_i = lift_{\Xi_i}(G_p)$ *for some* $G_p \sqsubseteq G$, $\langle E_{i1}, E_{i2}\rangle \Vdash \Xi_i \vdash G_u \sim G$, *and* $E_{12} \equiv E_{22}$, *then* $\langle E_{11}, E_{12}\rangle \mathbin{\text{;}} \langle E_1, E_1\rangle \iff \langle E_{21}, E_{22}\rangle \mathbin{\text{;}} \langle E_2, E_2\rangle$.

Proof. Note that by definition $E_1 \equiv E_2$. Also, $\forall \alpha^E \in FTN(E_i), E = lift_{\Xi_i}(\Xi_i(\alpha))$. Then we prove the $\Rightarrow$ direction (the other is analogous), by induction on the structure of the evidences $\langle E_{i1}, E_{i2}\rangle$. We skip cases where $E_i = ?$ or $E_{i1} = ?$, as the result is trivial (combination never fails).

Case ($\langle E_{11}, E_{12}\rangle = \langle E_{11}, \alpha^{E'_{12}}\rangle$). Then $\langle E_{21}, E_{22}\rangle = \langle E_{21}, \alpha^{E'_{22}}\rangle$, and $E_i = \langle \alpha^{E'_i}, \alpha^{E'_i}\rangle$, where $E'_i = lift_{\Xi_i}(\Xi_i(\alpha))$, and therefore $E'_{i2} \sqsubseteq E'_i$. And then by Lemma 6.30, the result holds immediately as both combinations are defined.

Case ($\langle E_{11}, E_{12}\rangle = \langle E_{11}, B\rangle$). Then $\langle E_{21}, E_{22}\rangle = \langle E_{12}, B\rangle$, and $\langle E_i, E_i\rangle = \langle B, B\rangle$, and the result trivially holds.

Case ($\langle E_{11}, E_{12}\rangle = \langle \alpha^{E'_{11}}, E_{12}\rangle$). The result holds by de inspection of consistent transitivity rule (sealR) and induction on evidence $\langle E'_{i1}, E_{i2}\rangle$.

Case ($\langle E_{11}, E_{12}\rangle = \langle E_{111} \to E_{112}, E_{121} \to E_{122}\rangle$). Then $\langle E_{11}, E_{12}\rangle = \langle E_{111} \to E_{112}, E_{121} \to E_{122}\rangle$, and $\langle E_i, E_i\rangle = \langle E'_{i1} \to E'_{i2}, E'_{i1} \to E'_{i2}\rangle$. As consistent transitivity is a symmetric relation, then the result holds by induction hypothesis on combinations of evidence $\langle E_{i11} \to E_{i12}\rangle \mathbin{\text{;}} \langle E'_{i1}, E'_{i1}\rangle$ and $\langle E_{i21} \to E_{i22}\rangle \mathbin{\text{;}} \langle E'_{i2}, E'_{i2}\rangle$.

For the other cases we proceed analogous to the function case. □

Proposition 6.14. *If* $(W, v_1, v_2) \in \mathcal{V}_\rho[\![G]\!]$ *and* $W' \succeq W$ *then* $(W, v_1, v_2) \in \mathcal{V}_\rho[\![G]\!]$.

Proposition 6.15 (**Compositionality**). *If*

- $W.\Xi_i(\alpha) = \rho(G')$ *and* $W.\kappa(\alpha) = \mathcal{V}_\rho[\![G']\!]$,
- $E'_i = lift_{W.\Xi_i}(\rho(G'))$,
- $E_i = lift_{W.\Xi_i}(G_p)$ *for some* $G_p \sqsubseteq \rho(G)$,
- $\rho' = \rho[X \mapsto \alpha]$,
- $\varepsilon_i = \langle E_i[\alpha^{E'_i}/X], E_i[E'_i/X]\rangle$, *such that* $\varepsilon_i \vdash W.\Xi_i \vdash \rho(G[\alpha/X]) \sim \rho(G[G'/X])$, *and*

- ${\varepsilon_i}^{-1} = \langle E_i[E'_i/X], E_i[\alpha^{E'_i}/X]\rangle$, *such that* ${\varepsilon_i}^{-1} \vdash W.\Xi_i \vdash \rho(G[G'/X]) \sim \rho(G[\alpha/X])$, *then*

(1)

$$(W, \varepsilon'_1 u_1 :: \rho'(G), \varepsilon'_2 u_2 :: \rho'(G)) \in \mathcal{V}_{\rho'}[\![G]\!] \Rightarrow$$
$$(W, \varepsilon_1(\varepsilon'_1 u_1 :: \rho(G)) :: \rho(G\,[G'/X]), \varepsilon_2(\varepsilon'_2 u_2 :: \rho(G)) :: \rho(G\,[G'/X])) \in \mathcal{T}_\rho[\![G\,[G'/X]]\!]$$

(2)

$$(W, \varepsilon'_1 u_1 :: \rho(G\,[G'/X]), \varepsilon'_2 u_2 :: \rho(G\,[G'/X])) \in \mathcal{V}_\rho[\![G\,[G'/X]]\!] \Rightarrow$$
$$(W, {\varepsilon_1}^{-1}(\varepsilon'_1 u_1 :: \rho(G\,[G'/X])) :: \rho'(G), {\varepsilon_2}^{-1}(\varepsilon'_2 u_2 :: \rho(G\,[G'/X])) :: \rho'(G)) \in \mathcal{T}_{\rho'}[\![G]\!]$$

Proof. We proceed by induction on G. Let suppose that $\varepsilon_1.n = k$, $\varepsilon_1^{-1}.n = l$ and $\varepsilon'_1.n = m$. Let $v_i = \varepsilon'_i u_i :: \rho'(G)$. We prove (1) first.

Case (**Type Variable X:** $G = X$). Let $v_i = \langle H_{i1}, \alpha^{E_{i2}}\rangle u_i :: \alpha$. Then we know that

$$(W, \langle H_{11}, \alpha^{E_{12}}\rangle u_1 :: \alpha, \langle H_{21}, \alpha^{E_{22}}\rangle u_2 :: \alpha) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![X]\!]$$

which is equivalent to

$$(W, \langle H_{11}, \alpha^{E_{12}}\rangle u_1 :: \alpha, \langle H_{21}, \alpha^{E_{22}}\rangle u_2 :: \alpha) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![\alpha]\!]$$

As $W.\Xi_i(\alpha) = \rho(G')$ and $W.\kappa(\alpha) = \mathcal{V}_\rho[\![G']\!]$, we know that:

$$(\downarrow_1 W, \langle H_{11}, E_{12}\rangle u_1 :: \rho(G'), \langle H_{21}, E_{22}\rangle u_2 :: \rho(G')) \in \mathcal{V}_\rho[\![G']\!]$$

Then $\varepsilon_i \vdash W.\Xi_i \vdash \alpha \sim \rho(G')$, and ε_i has to have the form $\varepsilon_i = \langle \alpha^{E'_i}, E'_i\rangle$. As $E'_i = \mathit{lift}_{W.\Xi_i}(\rho(G'))$ (initial evidence for α), then $E_{i2} \sqsubseteq E'_i$, and therefore by Lemma 6.30: $\langle H_{i1}, \alpha^{E_{i2}}\rangle \circ\!\!\!\!\,; \langle \alpha^{E'_i}, E'_i\rangle = \langle H_{i1}, E_{i2}\rangle$, and then we have to prove that

$$(\downarrow_k W, \langle H_{11}, E_{12}\rangle u_1 :: \rho(G'), \langle H_{21}, E_{22}\rangle u_2 :: \rho(G')) \in \mathcal{V}_\rho[\![G']\!]$$

which follow by Lemma 6.14 and the fact that $k > 0$.

Case (**Type Variable Y:** $G = Y$). Let $v_i = \langle H_{i1}, \beta^{E_{i2}}\rangle u_i :: \beta$, where $\rho'(Y) = \beta$. Then we know that

$$(W, \langle H_{11}, \beta^{E_{12}}\rangle u_1 :: \beta, \langle H_{21}, \beta^{E_{22}}\rangle u_2 :: \beta) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![Y]\!]$$

which is equivalent to

$$(W, \langle H_{11}, \beta^{E_{12}}\rangle u_1 :: \beta, \langle H_{21}, \beta^{E_{22}}\rangle u_2 :: \beta) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![\beta]\!]$$

Then $\varepsilon_i \vdash W.\Xi_i \vdash \beta \sim \beta$, and ε_i has to have the form $\varepsilon_i = \langle \beta^{E'_i}, \beta^{E'_i}\rangle$, and $\beta^{E'_i} = \mathit{lift}_{W.\Xi_i}(\beta)$. By Lemma 6.13, we assume that both combinations of evidence are defined (otherwise the result holds immediately). Therefore, by Lemma 6.30, we know that

$$\langle H_{i1}, \beta^{E_{i2}}\rangle \circ\!\!\!\!\,; \langle \beta^{E'_i}, \beta^{E'_i}\rangle = \langle H_{i1}, \beta^{E_{i2}}\rangle$$

Then we have to prove that

$$(\downarrow_k W, \langle H_{11}, \beta^{E_{12}}\rangle u_1 :: \beta, \langle H_{21}, \beta^{E_{22}}\rangle u_2 :: \beta) \in \mathcal{V}_\rho[\![\beta]\!]$$

which follows Lemma 6.14.

Case (**Unknown Type:** $G = ?$). Let $v_i = \langle H_{i1}, E_{i2}\rangle u_i :: ?$. Then by definition of $\mathcal{V}_\rho[\![?]\!]$, let $G'' = \mathit{const}(E_{i2})$ (where $G'' \neq ?$). Then we know

$$(W, \langle H_{11}, E_{12}\rangle u_1 :: G'', \langle H_{21}, E_{22}\rangle u_2 :: G'') \in \mathcal{V}_\rho[\![G'']\!]$$

We are required to prove that:

$$(W, \varepsilon_1(\langle H_{11}, E_{12}\rangle u_1 :: ?) :: ?, \varepsilon_2(\langle H_{21}, E_{22}\rangle u_2 :: ?) :: ?) \in \mathcal{T}_\rho[\![?]\!]$$

If $\varepsilon_i = \langle ?, ? \rangle$, then, $\langle H_{i1}, E_{i2} \rangle \mathbin{;} \langle ?, ? \rangle = \langle H_{i1}, E_{i2} \rangle$, by Lemma 6.30, the result holds immediately.

If $\varepsilon_i \neq \langle ?, ? \rangle$. Then we proceed similar to the other cases where $G \neq ?$. Note that we know that

$$(W, \langle H_{11}, E_{12} \rangle u_1 :: G'', \langle H_{21}, E_{22} \rangle u_2 :: G'') \in \mathcal{V}_\rho[\![G'']\!]$$

where $G'' \neq ?$ and we are required to prove that

$$(W, \varepsilon_1(\langle H_{11}, E_{12} \rangle u_1 :: G'') :: G'', \varepsilon_2(\langle H_{21}, E_{22} \rangle u_2 :: G'')) \in \mathcal{V}_\rho[\![G'']\!]$$

Case (**Function Type:** $G = G_1 \to G_2$). We know that

$$(W, v_1, v_2) \in \mathcal{V}_{\rho'}[\![G_1 \to G_2]\!]$$

Then we have to prove that

$$\begin{aligned}(\downarrow_k W, &(\varepsilon_1' \mathbin{;} \varepsilon_1)(\lambda x : G_1'.t_1) :: \rho(G_1[G'/X]) \to \rho(G_2[G'/X]),\\ &(\varepsilon_2' \mathbin{;} \varepsilon_2)(\lambda x : G_2'.t_2) :: \rho(G_1[G'/X]) \to \rho(G_2[G'/X])) \in \mathcal{V}_\rho[\![G_1[G'/X] \to G_2[G'/X]]\!]\end{aligned}$$

Let us call $v_i'' = (\varepsilon_i' \mathbin{;} \varepsilon_i)(\lambda x : G_i'.t_i) :: \rho'(G_1) \to \rho'(G_2)$. By unfolding, we have to prove that

$$\forall W' \geq (\downarrow_k W).\forall v_1', v_2'.(\downarrow_1 W', v_1', v_2') \in \mathcal{V}_\rho[\![G_1[G'/X]]\!] \Rightarrow (W', v_1''\ v_1', v_2''\ v_2') \in \mathcal{T}_\rho[\![G_2[G'/X]]\!]$$

Suppose that $v_i' = \varepsilon_i'' u_i' :: \rho(G_1[G'/X])$, by inspection of the reduction rules, we know that

$$W'.\Xi_i \triangleright v_i''\ v_i' \longmapsto^* W'.\Xi_i \triangleright (cod(\varepsilon_i') \mathbin{;} cod(\varepsilon_i))t_i[(\varepsilon_i'' \mathbin{;} (dom(\varepsilon_i) \mathbin{;} dom(\varepsilon_i')))u_i' :: G_i')/x] :: \rho(G_2[G'/X]))$$

This is equivalent by Lemma 6.18,

$$W'.\Xi_i \triangleright v_i''\ v_i' \longmapsto^* W'.\Xi_i \triangleright (cod(\varepsilon_i') \mathbin{;} cod(\varepsilon_i))t_i[((\varepsilon_i'' \mathbin{;} dom(\varepsilon_i)) \mathbin{;} dom(\varepsilon_i'))u_i' :: G_i')/x] :: \rho(G_2[G'/X]))$$

Therefore, we know that

$$\begin{gathered}W'.\Xi_1 \triangleright v_1''\ v_1' \longmapsto^{m+k+1}\\ W'.\Xi_1 \triangleright (cod(\varepsilon_1') \mathbin{;} cod(\varepsilon_1))t_1[(\varepsilon_1'' \mathbin{;} (dom(\varepsilon_1) \mathbin{;} dom(\varepsilon_i')))u_1' :: G_i')/x] :: \rho(G_2[G'/X])) \longmapsto^{k^*}\\ \Xi_1 \triangleright (cod(\varepsilon_1') \mathbin{;} cod(\varepsilon_1))v_{1f} :: \rho(G_2[G'/X])) \longmapsto^{m+k}\\ \Xi_1 \triangleright v_1^*\end{gathered}$$

where $v_{1f} = \varepsilon_{1f} u_{1f} :: \rho'(G_2)$ and $v_1^* = \varepsilon_{1f} \mathbin{;} (cod(\varepsilon_1') \mathbin{;} cod(\varepsilon_1))u_{1f} :: \rho(G_2[G'/X])$.

Notice that $dom(\varepsilon_i) \vdash W.\Xi_i \vdash \rho(G_1[G'/X]) \sim \rho(G_1[\alpha/X])$, by Lemma 6.13, we assume that both combinations of evidence are defined (otherwise the result holds immediately) , then let us assume that $(\varepsilon_i'' \mathbin{;} dom(\varepsilon_i))$ is defined. We can use induction hypothesis on v_i', with evidences $dom(\varepsilon_i)$. Then we know that $(\downarrow_{k+1} W', (\varepsilon_1'' \mathbin{;} dom(\varepsilon_1))u_1' :: \rho'(G_1), (\varepsilon_2'' \mathbin{;} dom(\varepsilon_2))u_2' :: \rho'(G_1)) \in \mathcal{V}_{\rho'}[\![G_1]\!]$. Let us call $v_i''' = (\varepsilon_i'' \mathbin{;} dom(\varepsilon_i))u_i' :: \rho'(G_1)$.

Now we instantiate

$$(W, v_1, v_2) \in \mathcal{V}_{\rho'}[\![G_1 \to G_2]\!]$$

with $\downarrow_k W'$ and v_i''' and

$$(\downarrow_{k+1} W', (\varepsilon_1'' \mathbin{;} dom(\varepsilon_1))u_1' :: \rho'(G_1), (\varepsilon_2'' \mathbin{;} dom(\varepsilon_2))u_2' :: \rho'(G_1)) \in \mathcal{V}_{\rho'}[\![G_1]\!]$$

to obtain that either both executions reduce to an error (then the result holds immediately), or $\exists W'' \geq \downarrow_k W'$ such that $W''.j + 2m + 1 + k^* + k = W'.j$ and $(W'', v'_{f1}, v'_{f2}) \in \mathcal{V}_{\rho'}[\![G_2]\!]$

$$\begin{aligned}W'.\Xi_i \triangleright v_i\ v_i''' &\longmapsto^* W'.\Xi_i \triangleright cod(\varepsilon_i')t[((\varepsilon_i'' \mathbin{;} dom(\varepsilon_i)) \mathbin{;} dom(\varepsilon_i'))u_i' :: G_i')/x] :: \rho'(G_2))\\ &\longmapsto^* W''.\Xi_i \triangleright v'_{fi}\end{aligned}$$

Suppose that $v_{fi}' = \varepsilon_{fi}' u_{fi} :: \rho'(G_2)$.

Also, we know that

$$\begin{gathered}W'.\Xi_1 \triangleright v_1\ v_1''' \longmapsto^{m+1}\\ W'.\Xi_1 \triangleright cod(\varepsilon_1')t_1[(\varepsilon_1'' \mathbin{;} (dom(\varepsilon_1) \mathbin{;} dom(\varepsilon_i')))u_1' :: G_i')/x] :: \rho'(G_2) \longmapsto^{k^*}\\ \Xi_1 \triangleright cod(\varepsilon_1')v_{1f} :: \rho'(G_2) \longmapsto^{m}\\ \Xi_1 \triangleright v'_{1f}\end{gathered}$$

Then we use induction hypothesis once again using evidences $cod(\varepsilon_i)$ over v'_{if} (noticing that by Lemma 6.13, the combination of evidence either both fail or both are defined), to obtain that,

$$\begin{gathered}(\downarrow_k W'', (\varepsilon_{f1} \mathbin{;} cod(\varepsilon_1') \mathbin{;} cod(\varepsilon_1))u_{f1} :: \rho(G_2[G'/X]),\\ (\varepsilon_{f2} \mathbin{;} cod(\varepsilon_2') \mathbin{;} cod(\varepsilon_2))u_{f2} :: \rho(G_2[G'/X])) \in \mathcal{V}_\rho[\![G_2[G'/X]]\!]\end{gathered}$$

and the result holds. Note that $(\downarrow_k W'').j + 1 + 2m + 2k + k^* = W'.j$

Case (**Universal Type:** $\forall Y.G_1$). We know that

$$(W, v_1, v_2) \in \mathcal{V}_{\rho'}[\![\forall Y.G_1]\!]$$

Then we have to prove that

$$\begin{gathered}(\downarrow_k W, (\varepsilon_1' \mathbin{;} \varepsilon_1)(\Lambda Y.t_1) :: \forall Y.\rho(G_1[G'/X]),\\ (\varepsilon_2' \mathbin{;} \varepsilon_2)(\Lambda Y.t_2) :: \forall Y.\rho(G_1[G'/X])) \in \mathcal{V}_\rho[\![\forall Y.G_1[G'/X]]\!]\end{gathered}$$

Let $\varepsilon_i' = \langle \forall Y.E_{i1}, \forall Y.E_{i2}\rangle$ and $\varepsilon_i = \langle \forall Y.E_{i1}', \forall Y.E_{i2}'\rangle = \langle \forall Y.E_i''[\alpha^{E_i'}/X], \forall Y.E_i''[E_i'/X]\rangle$, where $E_i = \forall Y.E_i''$. Let us call $v_i'' = (\varepsilon_i' \mathbin{;} \varepsilon_i)(\Lambda Y.t_i) :: \forall Y.\rho(G_1[G'/X])$. By unfolding, we have to prove that

$$\begin{aligned}&\forall W' \geq (\downarrow_k W).\forall t_1'', t_2'', G_1', G_2', \beta, \varepsilon_1'', \varepsilon_2''.\forall R \in \textsc{Rel}_{W'.j}[G_1', G_2'].\\ &(W'.\Xi_1 \vdash G_1' \land W'.\Xi_2 \vdash G_2' \land\\ &W'.\Xi_1 \triangleright v_1''[G_1'] \longmapsto W'.\Xi_1, \beta := G_1' \triangleright \varepsilon_1'' t_1'' :: \rho(G_1)[G'/X][G_1'/Y] \land\\ &W'.\Xi_2 \triangleright v_2''[G_2'] \longmapsto W'.\Xi_2, \beta := G_2' \triangleright \varepsilon_2'' t_2'' :: \rho(G_1)[G'/X][G_2/Y]) \Rightarrow\\ &(W^*, t_1'', t_2'') \in \mathcal{T}_{\rho[Y\mapsto\beta]}[\![G_1[G'/X]]\!]\end{aligned}$$

where $E_i^* = lift_{W'.\Xi_i}(G_i')$ and $W^* = \downarrow(W' \boxtimes (\beta, G_1', G_2', R)$

By inspection of the reduction rules we know that

$$t_i'' = (\langle E_{i1}[\beta^{E_i^*}/Y], E_{i2}[\beta^{E_i^*}/Y]\rangle \mathbin{;} \langle E_i''[\alpha^{E_i'}/X][\beta^{E_i^*}/Y], E_i''[E_i'/X][\beta^{E_i^*}/Y]\rangle)t_i[\beta^{E_i^*}/Y] :: \rho(G_1[G'/X][\beta/Y])$$

Note that $(\langle E_{i1}[\beta^{E_i^*}/Y], E_{i2}[\beta^{E_i^*}/Y]\rangle \mathbin{;} \langle E_i''[\alpha^{E_i'}/X][\beta^{E_i^*}/Y], E_i''[E_i'/X][\beta^{E_i^*}/Y]\rangle).n = m + k$. Therefore, we know that

$$\begin{gathered}W^*.\Xi_1 \triangleright t_1'' \longmapsto^{k^*}\\ \Xi_1 \triangleright (\langle E_{i1}[\beta^{E_i^*}/Y], E_{i2}[\beta^{E_i^*}/Y]\rangle \mathbin{;} \langle E_i''[\alpha^{E_i'}/X][\beta^{E_i^*}/Y], E_i''[E_i'/X][\beta^{E_i^*}/Y]\rangle)\\ v_{m1} :: \rho(G_1[G'/X][\beta/Y]) \longmapsto^{k+m} \Xi_1 \triangleright v_1^*\end{gathered}$$

By the reduction rule of the type application we know that:

$$W'.\Xi_i \triangleright v_i[G_i'] \longmapsto W'.\Xi_i, \beta := G_i' \triangleright \langle E_i^\#[\beta^{E_i^*}/Y], E_i^\#[E_i^*/Y]\rangle t_i' :: \rho(G_1[G'/X][G_i'/Y])$$

where $t_i' = (\langle E_{i1}[\beta^{E_i^*}/Y], E_{i2}[\beta^{E_i^*}/Y]\rangle t_i[\beta^{E_i^*}/Y] :: \rho(G_1[G'/X][\beta/Y]))$. Now we instantiate

$$(W, v_1, v_2) \in \mathcal{V}_{\rho'}[\![\forall Y.G_1]\!]$$

with W', G_1', G_2', R, t_1', t_2', β, and evidences $\langle E_{i1}[\beta^{E_i^*}/Y], E_{i2}[E_i^*/Y]\rangle$, to obtain that

$$(W^*, t_1', t_2') \in \mathcal{T}_{\rho'[Y\mapsto\beta]}[\![G_1]\!]$$

then either both executions reduce to an error (then the result holds immediately), or $\exists W'' \geq W^*, v_{fi}$, such that $(W'', v_{f1}, v_{f2}) \in \mathcal{V}_{\rho'[Y\mapsto\beta]}[\![G_1]\!]$ and

$$\begin{aligned}&W^*.\Xi_i \triangleright (\langle E_{i1}[\beta^{E_i^*}/Y], E_{i2}[\beta^{E_i^*}/Y]\rangle t_i[\beta^{E_i^*}/Y] :: \rho'(G_1[\beta/Y]))\\ \longmapsto^* &W''.\Xi_i \triangleright (\langle E_{i1}[\beta^{E_i^*}/Y], E_{i2}[\beta^{E_i^*}/Y]\rangle v_{mi} :: \rho'(G_1[\beta/Y]))\\ \longmapsto &W''.\Xi_i \triangleright v_{fi}\end{aligned}$$

$$\begin{aligned}&W^*.\Xi_1 \triangleright (\langle E_{11}[\beta^{E_1^*}/Y], E_{i2}[\beta^{E_1^*}/Y]\rangle t_1[\beta^{E_1^*}/Y] :: \rho'(G_1[\beta/Y]))\\ \longmapsto^{k^*} &W''.\Xi_1 \triangleright (\langle E_{11}[\beta^{E_1^*}/Y], E_{i2}[\beta^{E_1^*}/Y]\rangle v_{m1} :: \rho'(G_1[\beta/Y]))\\ \longmapsto^{m} &W''.\Xi_1 \triangleright v_{f1}\end{aligned}$$

Suppose that $v_{fi} = (\varepsilon_{fi} \mathbin{⨾} \langle E_{i1}[\beta^{E_i^*}/Y], E_{i2}[\beta^{E_i^*}/Y]\rangle)u_{fi} :: \rho'(G_1[\beta/Y])$. As $E_{12}[\beta^{E_1^*}/Y] \equiv E_{22}[\beta^{E_2^*}/Y]$, then $unlift(E_{12}[\beta^{E_1^*}/Y]) = unlift(E_{22}[\beta^{E_2^*}/Y])$. Then we use induction hypothesis using $\rho'[Y \mapsto \beta]$, evidences $\langle E_i''[E_i^*/Y], E_i''[E_i^*/Y]\rangle$, where $E_i''[E_i^*/Y] = lift_{W''.\Xi_i}(unlift(E_{i2}[\beta^{E_i^*}/Y]))$ as $E_i = \forall Y.E_i''$,

$$\mathcal{I}(lift_{W''.\Xi_i}(G_1[\beta/Y]), lift_{W''.\Xi_i}(G_1[\beta/Y])) = \langle E_i''[E_i^*/Y], E_i''[E_i^*/Y]\rangle$$

also we know that:

$$\langle E_i''[E_i^*/Y][\alpha^{E_i'}/X], E_i''[E_i^*/Y][E_i'/X]\rangle = \langle E_i''[\alpha^{E_i'}/X][E_i^*/Y], E_i''[E_i'/X][E_i^*/Y]\rangle$$

Note that $\rho(G_1[\beta/Y]) = \rho[Y \mapsto \beta](G_1)$. Then we know that

$$\begin{aligned}(\downarrow_k W'',&((\varepsilon_{f1} \mathbin{⨾} \langle E_{11}[\beta^{E_1^*}/Y], E_{12}[\beta^{E_1^*}/Y]\rangle) \mathbin{⨾} \langle E_1''[\alpha^{E_1'}/X][E_1^*/Y], E_1''[E_1'/X][E_1^*/Y]\rangle)u_{f1} :: \rho[Y \mapsto \beta](G_1[G'/X]),\\ &((\varepsilon_{f2} \mathbin{⨾} \langle E_{21}[\beta^{E_2^*}/Y], E_{22}[\beta^{E_2^*}/Y]\rangle) \mathbin{⨾} \langle E_2''[\alpha^{E_2'}/X][E_2^*/Y], E_2''[E_2'/X][E_2^*/Y]\rangle)u_{f2} :: \rho[Y \mapsto \beta](G_1[G'/X]))\\ &\in \mathcal{V}_{\rho[Y\mapsto\beta]}[\![G_1[G'/X]]\!]\end{aligned}$$

then by inspection of the reduction rules:

$$\begin{aligned}&W^*.\Xi_i \triangleright t_i''\\ \longmapsto^* &W''.\Xi_i \triangleright ((\langle E_{i1}[\beta^{E_i^*}/Y], E_{i2}[\beta^{E_i^*}/Y]\rangle \mathbin{⨾} \langle E_i''[\alpha^{E_i'}/X][\beta^{E_i^*}/Y], E_i''[E_i'/X][\beta^{E_i^*}/Y]\rangle)v_{mi} :: \rho'(G_1[\beta/Y]))\\ \longmapsto &W''.\Xi_i \triangleright (\varepsilon_{fi} \mathbin{⨾} (\langle E_{i1}[\beta^{E_i^*}/Y], E_{i2}[\beta^{E_i^*}/Y]\rangle \mathbin{⨾} \langle E_i''[\alpha^{E_i'}/X][E_i^*/Y], E_i''[E_i'/X][E_i^*/Y]\rangle))u_{fi} :: \rho[Y \mapsto \beta](G_i[G'/X])\end{aligned}$$

and by Lemma 6.18, we know that those two values belong to the interpretation of $\mathcal{V}_{\rho[Y\mapsto\beta]}[\![G_1[G'/X]]\!]$, and the result holds. Note that $\downarrow_k W''.k + m + k^* = W^*$.

Case (**Pair Type:** $G_1 \times G_2$). Analogous to the function case.

Case (**Base Type:** B). Trivial.

Then we prove as (2):

Case (**Type Variable X:** $G = X$). Let $v_i = \langle H_{i1}, E_{i2}\rangle u_i :: X[G'/X] = \langle H_{i1}, E_{i2}\rangle u_i :: G'$. Then we know that

$$(W, \langle H_{11}, E_{12}\rangle u_1 :: G', \langle H_{21}, E_{22}\rangle u_2 :: G') \in \mathcal{V}_\rho[\![G']\!]$$

and $\varepsilon_i{}^{-1} = \langle E_i', \alpha^{E_i'}\rangle$. Then we have to prove that

$$(\downarrow_l W, (\langle H_{11}, E_{12}\rangle \mathbin{⨾} \langle E_1', \alpha^{E_1'}\rangle)u_1 :: \alpha, (\langle H_{21}, E_{22}\rangle \mathbin{⨾} \langle E_2', \alpha^{E_2'}\rangle)u_2 :: \alpha) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![\alpha]\!]$$

By Lemma 6.13, we assume that both combinations of evidence are defined (otherwise the result holds immediately). Then by definition of transitivity and Lemma 6.30, we know that $(\langle H_{i1}, E_{i2}\rangle \mathbin{⨾} \langle E_i', \alpha^{E_i'}\rangle) = \langle H_{i1}, \alpha^{E_{i2}}\rangle$. Then we have to prove that

$$(\downarrow_l W, \langle H_{11}, \alpha^{E_{12}}\rangle u_1 :: \alpha, \langle H_{21}, \alpha^{E_{22}}\rangle u_2 :: \alpha) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![\alpha]\!]$$

but as α is sync, then that is equivalent to

$$(\downarrow_{l-1} W, \langle H_{11}, E_{12}\rangle u_1 :: G', \langle H_{21}, E_{22}\rangle u_2 :: G') \in \mathcal{V}_\rho[\![G']\!]$$

which follows by the premise and Lemma 6.14.

Also, we have to prove that ($\forall \Xi', \varepsilon', G^*$ such that $(\downarrow_{l-1} W \in \mathcal{S}[\![\Xi']\!] \wedge \varepsilon' \vdash \Xi' \vdash \alpha \sim G^*)$, we get that

$$(\downarrow_{l-1} W, \varepsilon'(\langle H_{11}, \alpha^{E_{12}}\rangle u_1 :: \alpha) :: G^*, \varepsilon'(\langle H_{21}, \alpha^{E_{22}}\rangle u_2 :: \alpha) :: G^*) \in \mathcal{T}_\rho[\![G^*]\!])$$

or what is the same (($\langle H_{11}, \alpha^{E_{12}}\rangle \operatorname{\circ\!\!\!\!,} \varepsilon'$) fails the result follows immediately)

$$(\downarrow_{l-1-k'} W, (\langle H_{11}, \alpha^{E_{12}}\rangle \operatorname{\circ\!\!\!\!,} \varepsilon')u_1 :: G^*, (\langle H_{21}, \alpha^{E_{22}}\rangle \operatorname{\circ\!\!\!\!,} \varepsilon')u_2 :: G^*) \in \mathcal{V}_\rho[\![G^*]\!])$$

where $\varepsilon' = \langle \alpha^{E_1^*}, E_2^*\rangle$ and $\varepsilon'.n = k'$. By definition of transitivity and Lemma 6.30, we know that

$$\langle H_{i1}, \alpha^{E_{i2}}\rangle \operatorname{\circ\!\!\!\!,} \langle \alpha^{E_1^*}, E_2^*\rangle = \langle H_{i1}, E_{i2}\rangle \operatorname{\circ\!\!\!\!,} \langle E_1^*, E_2^*\rangle$$

We know that $\langle E_1^*, E_2^*\rangle \vdash \Xi' \vdash G' \sim G^*$. Since $\langle E_1^*, E_2^*\rangle \vdash \Xi \vdash G' \sim G^*$, $\downarrow_{l-1} W \in \mathcal{S}[\![\Xi']\!]$, $(\downarrow_{l-1} W, \langle H_{11}, E_{12}\rangle u_1 :: G', \langle H_{21}, E_{22}\rangle u_2 :: G') \in \mathcal{V}_\rho[\![G']\!]$, by Lemma 6.17, we know that (since $(\langle H_{11}, \alpha^{E_{12}}\rangle \operatorname{\circ\!\!\!\!,} \varepsilon')$ does not fail then $(\langle H_{11}, E_{12}\rangle \operatorname{\circ\!\!\!\!,} \langle E_1^*, E_2^*\rangle)$ also does not fail by the transitivity rules)

$$(\downarrow_{l-1-k'} W, (\langle H_{11}, E_{12}\rangle \operatorname{\circ\!\!\!\!,} \langle E_1^*, E_2^*\rangle)u_1 :: G^*, (\langle H_{21}, E_{22}\rangle \operatorname{\circ\!\!\!\!,} \langle E_1^*, E_2^*\rangle)u_2 :: G^*) \in \mathcal{V}_\rho[\![G^*]\!])$$

The result follows immediately.

Case (**Type Variable Y:** $G = Y$). Let $v_i = \langle H_{i1}, \beta^{E_{i2}}\rangle u_i :: \rho(Y[G'/X]) = \langle H_{i1}, \beta^{E_{i2}}\rangle u_i :: \beta$ (where $\rho(Y) = \beta$). Then we know that

$$(W, \langle H_{11}, \beta^{E_{12}}\rangle u_1 :: \beta, \langle H_{21}, \beta^{E_{22}}\rangle u_2 :: \beta) \in \mathcal{V}_\rho[\![\beta]\!]$$

We know that ${\varepsilon_i}^{-1} \vdash W.\Xi_i \vdash \beta \sim \beta$, therefore ${\varepsilon_i}^{-1}$ has to have the form ${\varepsilon_i}^{-1} = \langle \beta^{E'_i}, \beta^{E'_i}\rangle = \mathcal{I}(lift_{W.\Xi_i}(\beta), lift_{W.\Xi_i}(\beta))$. As ${\varepsilon_i}^{-1}$ is the initial evidence for β, then $E_{i2} \sqsubseteq E'_i$, and therefore by definition of the transitivity and Lemma 6.30:

$$\langle H_{i1}, \beta^{E_{i2}}\rangle \operatorname{\circ\!\!\!\!,} \langle \beta^{E'_i}, \beta^{E'_i}\rangle = \langle H_{i1}, \beta^{E_{i2}}\rangle$$

Then we have to prove that:

$$(\downarrow_l W, (\langle H_{11}, \beta^{E_{12}}\rangle \operatorname{\circ\!\!\!\!,} \langle \beta^{E'_1}, \beta^{E'_1}\rangle)u_1 :: \beta, (\langle H_{21}, \beta^{E_{22}}\rangle \operatorname{\circ\!\!\!\!,} \langle E'_2, \beta^{E'_2}\rangle)u_2 :: \beta) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![\beta]\!]$$

or what is the same

$$(\downarrow_l W, \langle H_{11}, \beta^{E_{12}}\rangle u_1 :: \beta, \langle H_{21}, \beta^{E_{22}}\rangle u_2 :: \beta) \in \mathcal{V}_\rho[\![\beta]\!]$$

which follows by the premise and Lemma 6.14.

Case (**Unknown Type:** $G = ?$). Let $v_i = \langle H_{i1}, E_{i2}\rangle u_i :: ?$. Then by definition of $\mathcal{V}_\rho[\![?]\!]$, let $G'' = const(E_{i2})$ (where $G'' \neq ?$). Then we know

$$(W, \langle H_{11}, E_{12}\rangle u_1 :: G'', \langle H_{21}, E_{22}\rangle u_2 :: G'') \in \mathcal{V}_\rho[\![G'']\!]$$

If ${\varepsilon_i}^{-1} = \langle ?, ?\rangle$, then, $\langle H_{i1}, E_{i2}\rangle \operatorname{\circ\!\!\!\!,} \langle ?, ?\rangle = \langle H_{i1}, E_{i2}\rangle$, by Lemma 6.30, the result holds immediately.

If ${\varepsilon_i}^{-1} \neq \langle ?, ?\rangle$. Then we proceed similar to the other cases where $G \neq ?$. Note that we know that

$$(W, \langle H_{11}, E_{12}\rangle u_1 :: G'', \langle H_{21}, E_{22}\rangle u_2 :: G'') \in \mathcal{V}_\rho[\![G'']\!]$$

where $G'' \neq ?$ and we are required to prove that

$$(W, \varepsilon_1(\langle H_{11}, E_{12}\rangle u_1 :: G'') :: G'', \varepsilon_2(\langle H_{21}, E_{22}\rangle u_2 :: G'')) \in \mathcal{V}_\rho[\![G'']\!]$$

Case (**Function Type:** $G = G_1 \to G_2$). Let $v_i = \varepsilon'_i(\lambda x : G'_i.t_i) :: \rho(G[G'/X])$ We know that

$$(W, v_1, v_2) \in \mathcal{V}_\rho[\![G_1[G'/X] \to G_2[G'/X]]\!]$$

Then we have to prove that

$$(\downarrow_l W, (\varepsilon'_1 \mathbin{⨟} \varepsilon_1{}^{-1})(\lambda x : G'_1.t_1) :: \rho'(G_1) \to \rho'(G_2),\\ (\varepsilon'_2 \mathbin{⨟} \varepsilon_2{}^{-1})(\lambda x : G'_2.t_2) :: \rho'(G_1) \to \rho'(G_2)) \in \mathcal{V}_{\rho'}[\![G_1 \to G_2]\!]$$

Let us call $v''_i = (\varepsilon'_i \mathbin{⨟} \varepsilon_i{}^{-1})(\lambda x : G'_i.t_i) :: \rho'(G_1) \to \rho'(G_2)$. By unfolding, we have to prove that

$$\forall W' \geq (\downarrow_l W).\forall v'_1, v'_2.(\downarrow_1 W', v'_1, v'_2) \in \mathcal{V}_{\rho'}[\![G_1]\!] \Rightarrow (W', v''_1\ v'_1, v''_2\ v'_2) \in \mathcal{T}_{\rho'}[\![G_2]\!]$$

Suppose that $v'_i = \varepsilon''_i u'_i :: \rho'(G_1)$, by inspection of the reduction rules, we know that

$$W'.\Xi_i \triangleright v''_i\ v'_i \longmapsto^* W'.\Xi_i \triangleright (cod(\varepsilon'_i) \mathbin{⨟} cod(\varepsilon_i{}^{-1}))t_i[(\varepsilon''_i \mathbin{⨟} (dom(\varepsilon_i{}^{-1}) \mathbin{⨟} dom(\varepsilon'_i))u'_i :: G'_i)/x] :: \rho'(G_2))$$

This is equivalent by Lemma 6.18,

$$W'.\Xi_i \triangleright v''_i\ v'_i \longmapsto^* W'.\Xi_i \triangleright (cod(\varepsilon'_i) \mathbin{⨟} cod(\varepsilon_i{}^{-1}))t_i[((\varepsilon''_i \mathbin{⨟} dom(\varepsilon_i{}^{-1})) \mathbin{⨟} dom(\varepsilon'_i))u'_i :: G'_i)/x] :: \rho'(G_2))$$

Also, we know that

$$W'.\Xi_1 \triangleright v''_1\ v'_1 \longmapsto^{l+m+1}$$
$$W'.\Xi_1 \triangleright (cod(\varepsilon'_1) \mathbin{⨟} cod(\varepsilon_1{}^{-1}))t_1[((\varepsilon''_1 \mathbin{⨟} dom(\varepsilon_1{}^{-1})) \mathbin{⨟} dom(\varepsilon'_1))u'_1 :: G'_1)/x] :: \rho'(G_2)) \longmapsto^{k^*}$$
$$\Xi_1 \triangleright (cod(\varepsilon'_1) \mathbin{⨟} cod(\varepsilon_1{}^{-1}))v_{1f} :: \rho'(G_2)) \longmapsto^{l+m}$$
$$\Xi_1 \triangleright v_1^*$$

where $v_{1f} = \varepsilon_{1f} u_{1f} :: \rho(G_2[G'/X])$ and $v_1^* = (\varepsilon_{1f} \mathbin{⨟} cod(\varepsilon'_1) \mathbin{⨟} cod(\varepsilon_1{}^{-1}))u_{1f} :: \rho'(G_2)$.

Notice that $dom(\varepsilon_i{}^{-1}) \vdash W.\Xi_i \vdash \rho(G_1[\alpha/X]) \sim \rho(G_1[G'/X])$, and as $dom(\varepsilon_i{}^{-1})$ is constructed using the interior (and thus $\pi_2(\varepsilon''_i) \sqsubseteq \pi_1(dom(\varepsilon_i{}^{-1}))$), then by definition of evidence $(\varepsilon''_i \mathbin{⨟} dom(\varepsilon_i{}^{-1}))$ is always defined. We can use induction hypothesis on v'_i, with evidences $dom(\varepsilon_i{}^{-1})$.

Then we know that

$$(\downarrow_{l+1} W', (\varepsilon''_1 \mathbin{⨟} dom(\varepsilon_1{}^{-1}))u'_1 :: \rho(G_1[G'/X]), (\varepsilon''_2 \mathbin{⨟} dom(\varepsilon_2{}^{-1}))u'_2 :: \rho(G_1[G'/X])) \in \mathcal{V}_\rho[\![G_1[G'/X]]\!]$$

Let us call $v'''_i = (\varepsilon''_i \mathbin{⨟} dom(\varepsilon_i{}^{-1}))u'_i :: \rho(G_1[G'/X])$.

Now we instantiate

$$(W, v_1, v_2) \in \mathcal{V}_\rho[\![G_1[G'/X] \to G_2[G'/X]]\!]$$

with $(\downarrow_l W')$ and v'''_i, to obtain that either both executions reduce to an error (then the result holds immediately), or $\exists W'' \geq (\downarrow_l W')$ such that $(W'', v'_{f1}, v'_{f2}) \in \mathcal{V}_\rho[\![G_2[G'/X]]\!]$, $W''.j + 2m + k^* = (\downarrow_l W').j$ $(W''.j + 1 + l + 2m + k^* = W'.j)$ and

$$W'.\Xi_i \triangleright v_i\ v'''_i \longmapsto W'.\Xi_i \triangleright cod(\varepsilon'_i)t_i[((\varepsilon''_i \mathbin{⨟} dom(\varepsilon_i{}^{-1})) \mathbin{⨟} dom(\varepsilon'_i))u'_i :: G'_i)/x] :: \rho(G_2[G'/X]))$$
$$\longmapsto^* W''.\Xi_i \triangleright v'_{fi}$$

Therefore, we know that

$$W'.\Xi_1 \triangleright v_1\ v'''_1 \longmapsto^{m+1}$$
$$W'.\Xi_1 \triangleright cod(\varepsilon'_1)t_1[((\varepsilon''_1 \mathbin{⨟} dom(\varepsilon_1{}^{-1})) \mathbin{⨟} dom(\varepsilon'_1))u'_1 :: G'_1)/x] :: \rho(G_2[G'/X])) \longmapsto^{k^*}$$
$$\Xi_1 \triangleright cod(\varepsilon'_1)v_{f1} :: \rho(G_2[G'/X])) \longmapsto^m$$
$$W''.\Xi_1 \triangleright v'_{f1}$$

Suppose that $v'_{fi} = \varepsilon'_{fi} u_{fi} :: \rho(G_2[G'/X])$ and $\varepsilon'_{f1} = \varepsilon_{f1} \mathbin{⨟} cod(\varepsilon'_1)$. Then we use induction hypothesis once again using evidences $cod(\varepsilon_i{}^{-1})$ and $(W'', v'_{f1}, v'_{f2}) \in \mathcal{V}_\rho[\![G_2[G'/X]]\!]$, (noticing that the combination of evidence does not fail as the evidence is obtained via the interior function i.e. the less precise evidence possible), to obtain that,

$$(\downarrow_l W'', (\varepsilon_{f1} \fatsemi cod(\varepsilon_1') \fatsemi cod(\varepsilon_1{}^{-1}))u_{f1} :: \rho'(G_2), (\varepsilon_{f2} \fatsemi cod(\varepsilon_2') \fatsemi cod(\varepsilon_2{}^{-1}))u_{f2} :: \rho'(G_2)) \in \mathcal{V}_{\rho'}[\![G_2]\!]$$

Note that $(\downarrow_l W'').j + 1 + 2l + 2m + k^* = W'.j$, and the result holds.
The remaining cases are similar.

□

Lemma 10.4 (**Compositionality**). *If*

- $W.\Xi_i(\alpha) = \rho(G')$ *and* $W.\kappa(\alpha) = \mathcal{V}_\rho[\![G']\!]$,
- $E_i' = lift_{W.\Xi_i}(\rho(G'))$,
- $E_i = lift_{W.\Xi_i}(G_p)$ *for some* $G_p \sqsubseteq \rho(G)$,
- $\rho' = \rho[X \mapsto \alpha]$,
- $\varepsilon_i = \langle E_i[\alpha^{E_i'}/X], E_i[E_i'/X]\rangle$, *such that* $\varepsilon_i \vdash W.\Xi_i \vdash \rho(G[\alpha/X]) \sim \rho(G[G'/X])$, *and*
- $\varepsilon_i{}^{-1} = \langle E_i[E_i'/X], E_i[\alpha^{E_i'}/X]\rangle$, *such that* $\varepsilon_i{}^{-1} \vdash W.\Xi_i \vdash \rho(G[G'/X]) \sim \rho(G[\alpha/X])$, *then*

(1) $(W, v_1, v_2) \in \mathcal{V}_{\rho'}[\![G]\!] \Rightarrow (W, \varepsilon_1 v_1 :: \rho(G\,[G'/X]), \varepsilon_2 v_2 :: \rho(G\,[G'/X])) \in \mathcal{T}_\rho[\![G\,[G'/X]]\!]$
(2) $(W, v_1, v_2) \in \mathcal{V}_\rho[\![G\,[G'/X]]\!] \Rightarrow (W, \varepsilon_1{}^{-1} v_1 :: \rho'(G), \varepsilon_2{}^{-1} v_2 :: \rho'(G)) \in \mathcal{T}_{\rho'}[\![G]\!]$

Proof. Direct by Prop. 10.4. □

Definition 6.16. $\rho \vdash \varepsilon_1 \equiv \varepsilon_2$ if $unlift(\pi_2(\varepsilon_1)) = unlift(\pi_2(\varepsilon_2))$

Proposition 6.17. *If*

- $(W, v_1, v_2) \in \mathcal{V}_\rho[\![G]\!]$
- $\varepsilon \Vdash \Xi; \Delta \vdash G \sim G'$
- $W \in \mathcal{S}[\![\Xi]\!]$ *and* $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$
- $\forall \alpha \in dom(\Xi).sync(\alpha, W)$

then:

$$(W, \rho_1(\varepsilon)v_1 :: \rho(G'), \rho_2(\varepsilon)v_2 :: \rho(G')) \in \mathcal{T}_\rho[\![G']\!]$$

where $sync(\alpha, W) \iff W.\Xi_1(\alpha) = W.\Xi_2(\alpha) \wedge W.\kappa(\alpha) = \lfloor \mathcal{V}_\emptyset[\![W.\Xi_i(\alpha)]\!] \rfloor_{W.j}$.

Proof. We proceed by induction on G and $W.j$. We know that $u_i \in G_i$ for some G_i, notice that $G_i \in$ HeadType $\cup$ TypeVar. In every case we apply Lemma 6.26 to show that $(\varepsilon_1 \fatsemi \varepsilon_1^\rho) \iff (\varepsilon_2 \fatsemi \varepsilon_2^\rho)$, so in all cases we assume that the transitivity does not fail (otherwise the proof holds immediately). Let us call $\varepsilon_1^\rho = \rho_1(\varepsilon)$ and $\varepsilon_2^\rho = \rho_2(\varepsilon)$. Let's suppose that $\varepsilon_1^\rho.n = k$ and $\varepsilon_1.n = l$.

Case (**Base type:** $G = B$ and $G' = B$). We know that v_i has the form $\langle B, B\rangle\, u :: B$, and we know that $(W, \langle B, B\rangle\, u :: B, \langle B, B\rangle\, u :: B) \in \mathcal{V}_\rho[\![B]\!]$. Also as $\varepsilon \vdash \Xi; \Delta \vdash B \sim B$, then $\varepsilon = \langle B, B\rangle$, then as $\rho_i(B) = B$, $\varepsilon_i \fatsemi \rho_i(\varepsilon) = \varepsilon_i$, and we have to prove that $(\downarrow_k W, \langle B, B\rangle\, u :: B, \langle B, B\rangle\, u :: B) \in \mathcal{V}_\rho[\![B]\!]$, which follows immediately because the premise and Lemma 6.14.

Case (**Function type:** $G = G_1'' \to G_2''$, and $G' = G_1' \to G_2'$). We know that:

$$(W, v_1, v_2) \in \mathcal{V}_\rho[\![G_1'' \to G_2'']\!]$$

Where $v_i = \varepsilon_i(\lambda x : G_{1i}.t_i) :: \rho(G_1'' \to G_2'')$ and $\varepsilon_i \vdash W.\Xi_i \vdash G_i \sim \rho(G_1'' \to G_2'')$.
We have to prove that:

$$(W, \varepsilon_1^\rho v_1 :: \rho(G_1' \to G_2'), \varepsilon_2^\rho v_2 :: \rho(G_1' \to G_2')) \in \mathcal{T}_\rho[\![G_1' \to G_2']\!]$$

Or what is the same:

$$(\downarrow_l W, (\varepsilon_1 \fatsemi \varepsilon_1^\rho)(\lambda x : G_{11}.t_1) :: \rho(G_1' \to G_2'), (\varepsilon_2 \fatsemi \varepsilon_2^\rho)(\lambda x : G_{12}.t_2) :: \rho(G_1' \to G_2')) \in \mathcal{T}_\rho[\![G_1' \to G_2']\!]$$

First we suppose that $(\varepsilon_i \fatsemi \varepsilon_i^\rho)$ does not fail and $(\varepsilon_i \fatsemi \varepsilon_i^\rho).n = k + l$, then we have to prove that:

$$\forall W' \geq \downarrow_l W. \forall v_1', v_2'. (\downarrow_1 W', v_1', v_2') \in \mathcal{V}_\rho[\![G_1']\!] \Rightarrow$$

$$(W', [(\varepsilon_1 \fatsemi \varepsilon_1^\rho)(\lambda x : G_{11}.t_1) :: \rho(G_1' \to G_2')]\ v_1', [(\varepsilon_2 \fatsemi \varepsilon_2^\rho)(\lambda x : G_{12}.t_2) :: \rho(G_1' \to G_2')]\ v_2') \in \mathcal{T}_\rho[\![G_2']\!]$$

where $v_i' = \varepsilon_i' u_i' :: \rho(G_1')$. Note that by the reduction rule of application terms, we obtain that:

$$W'.\Xi_i \triangleright ((\varepsilon_i \fatsemi \varepsilon_i^\rho)(\lambda x : G_{1i}.t_i) :: \rho(G_1' \to G_2')\ (\varepsilon_i' u_i' :: \rho(G_1') \longrightarrow^*$$

$$W'.\Xi_i \triangleright cod(\varepsilon_i \fatsemi \varepsilon_i^\rho)([(\varepsilon_i' \fatsemi dom(\varepsilon_i \fatsemi \varepsilon_i^\rho))u_i' :: G_{1i})/x]t_i) :: \rho(G_2')$$

We know by the Proposition 6.20 that $dom(\varepsilon_i \fatsemi \varepsilon_i^\rho) = dom(\varepsilon_i^\rho) \fatsemi dom(\varepsilon_i)$. Then by the Proposition 6.18 we know that:

$$\varepsilon_i' \fatsemi (dom(\varepsilon_i \fatsemi \varepsilon_i^\rho)) = \varepsilon_i' \fatsemi (dom(\varepsilon_i^\rho) \fatsemi dom(\varepsilon_i)) = (\varepsilon_i' \fatsemi dom(\varepsilon_i^\rho)) \fatsemi dom(\varepsilon_i)$$

Also, by the Proposition 6.21 it is follows that: $cod(\varepsilon_i \fatsemi \varepsilon_i^\rho) = cod(\varepsilon_i) \fatsemi cod(\varepsilon_i^\rho)$.

Then the following result is true:

$$W'.\Xi_i \triangleright cod(\varepsilon_i \fatsemi \varepsilon_i^\rho)([(\varepsilon_i' \fatsemi dom(\varepsilon_i \fatsemi \varepsilon_i^\rho))u_i' :: G_{1i})/x]t_i) :: \rho(G_2') =$$

$$W'.\Xi_i \triangleright cod((\varepsilon_i) \fatsemi cod(\varepsilon_i^\rho))([((\varepsilon_i' \fatsemi dom(\varepsilon_i^\rho)) \fatsemi dom(\varepsilon_i))u_i' :: G_{1i})/x]t_i) :: \rho(G_2')$$

So, we know that:

$$W'.\Xi_1 \triangleright ((\varepsilon_1 \fatsemi \varepsilon_i^\rho)(\lambda x : G_{11}.t_1) :: \rho(G_1' \to G_2')\ (\varepsilon_1' u_1' :: \rho(G_1') \longrightarrow^{l+k+1}$$

$$W'.\Xi_1 \triangleright cod(\varepsilon_1 \fatsemi \varepsilon_i^\rho)([(\varepsilon_1' \fatsemi dom(\varepsilon_1 \fatsemi \varepsilon_i^\rho))u_1' :: G_{11})/x]t_1) :: \rho(G_2') =$$

$$W'.\Xi_1 \triangleright cod((\varepsilon_1) \fatsemi cod(\varepsilon_i^\rho))([((\varepsilon_1' \fatsemi dom(\varepsilon_i^\rho)) \fatsemi dom(\varepsilon_1))u_1' :: G_{11})/x]t_1) :: \rho(G_2') \longrightarrow^{k^*}$$

$$\Xi_1 \triangleright (cod(\varepsilon_1) \fatsemi cod(\varepsilon_i^\rho))v_1^* :: \rho(G_2') \longrightarrow^{l+k}$$

$$\Xi_1 \triangleright (\varepsilon_1'' \fatsemi (cod(\varepsilon_1) \fatsemi cod(\varepsilon_i^\rho)))u_{1f} :: \rho(G_2')$$

where $v_1^* = \varepsilon_1'' u_{1f} :: \rho(G_2'')$ and $v_{1f} = (\varepsilon_1'' \fatsemi (cod(\varepsilon_1) \fatsemi cod(\varepsilon_i^\rho)))u_{1f} :: \rho(G_2')$.

We instantiate the induction hypothesis in $(\downarrow_1 W', v_1', v_2') \in \mathcal{V}_\rho[\![G_1']\!]$ with the type G_1'' and the evidences $dom(\varepsilon) \vdash \Xi; \Delta \vdash G_1' \sim G_1''$, where $dom(\varepsilon).n = l$. We obtain that:

$$(\downarrow_1 W', dom(\varepsilon_1^\rho)v_1' :: G_1'', dom(\varepsilon_2^\rho)v_2' :: G_1'') \in \mathcal{T}_\rho[\![G_1'']\!]$$

In particular we focus on a pair of values such that $(\varepsilon_i' \fatsemi dom(\varepsilon_i^\rho))$ does not fail (otherwise the result follows immediately). Then it is true that:

$$(\downarrow_{l+1} W', (\varepsilon_1' \fatsemi dom(\varepsilon_1^\rho))u_1' :: G_1'', (\varepsilon_2' \fatsemi dom(\varepsilon_2^\rho))u_2' :: G_1'') \in \mathcal{V}_\rho[\![G_1'']\!]$$

By the definition of $\mathcal{V}_\rho[\![G_1'' \to G_2'']\!]$ we know that:

$$\forall W'' \geq W. \forall v_1'', v_2''. (\downarrow_1 W'', v_1'', v_2'') \in \mathcal{V}_\rho[\![G_1'']\!] \Rightarrow (W'', v_1\ v_1'', v_2\ v_2'') \in \mathcal{T}_\rho[\![G_2'']\!]$$

We instantiate $v_i'' = (\varepsilon_i' \fatsemi dom(\varepsilon_i^\rho))u_i' :: \rho(G_1'')$ and $W'' = \downarrow_l W'$. Then we obtain that:

$$(\downarrow_l W', ((\varepsilon_1(\lambda x : G_{11}.t_1) :: \rho(G_1'' \to G_2''))\ ((\varepsilon_1' \fatsemi dom(\varepsilon_i^\rho))u_i' :: \rho(G_1'')),$$
$$(\varepsilon_2(\lambda x : G_{12}.t_2) :: \rho(G_1'' \to G_2''))\ ((\varepsilon_2' \fatsemi dom(\varepsilon_i^\rho))u_i' :: \rho(G_1''))) \in \mathcal{T}_\rho[\![G_2'']\!]$$

Then by Lemma 6.18, as $(\varepsilon_1' \fatsemi dom(\varepsilon_1^\rho)) \fatsemi dom(\varepsilon_1) = \varepsilon_1' \fatsemi (dom(\varepsilon_1^\rho)) \fatsemi dom(\varepsilon_1))$, then if $(dom(\varepsilon_1^\rho)) \fatsemi dom(\varepsilon_1))$ is not defined and $(dom(\varepsilon_2^\rho)) \fatsemi dom(\varepsilon_2))$ is defined, we get a contradiction as both must behave uniformly as the terms belong to $\mathcal{T}_\rho[\![G_2'']\!]$. Then if both combination of evidence fail, then the result follows immediately. Let us suppose that the combination does not fail, then

$$W'.\Xi_i \triangleright (\varepsilon_i(\lambda x : G_{1i}.t_i) :: \rho(G_1'' \to G_2''))\ ((\varepsilon_i' \mathbin{⨾} dom(\varepsilon_i^\rho))u_i' :: \rho(G_1'')) \longrightarrow^*$$
$$W'.\Xi_i \triangleright cod(\varepsilon_i)([((\varepsilon_i' \mathbin{⨾} dom(\varepsilon_i^\rho)) \mathbin{⨾} dom(\varepsilon_i))u_i' :: G_{1i})/x]t_i) :: \rho(G_2'')$$

So, we know that:

$$W'.\Xi_1 \triangleright ((\varepsilon_1(\lambda x : G_{11}.t_1) :: \rho(G_1'' \to G_2''))\ ((\varepsilon_1' \mathbin{⨾} dom(\varepsilon_i^\rho))u_i' :: \rho(G_1'')) \longrightarrow^{k+1}$$
$$W'.\Xi_1 \triangleright cod(\varepsilon_1)([(\varepsilon_1' \mathbin{⨾} dom(\varepsilon_i^\rho) \mathbin{⨾} dom(\varepsilon_1))u_1' :: G_{11})/x]t_1) :: \rho(G_2') \longrightarrow^{k^*}$$
$$\Xi_1 \triangleright cod(\varepsilon_1)v_1^* :: \rho(G_2') \longrightarrow^{k}$$
$$\Xi_1 \triangleright (\varepsilon_1'' \mathbin{⨾} cod(\varepsilon_1))u_{1f} :: \rho(G_2')$$

where $v'^*_1 = (\varepsilon_1'' \mathbin{⨾} cod(\varepsilon_1))u_{1f} :: \rho(G_2')$.

Thus, we know that $\exists W''' \geq\downarrow_l W'$ such that $(W''', v'^*_1, v'^*_2) \in \mathcal{V}_\rho[\![G_2'']\!]$, $W'''.\Xi_1 = \Xi_1$ and $W'''.j + 1 + 2k + k^* = (\downarrow_l W').j$, or what is the same $W'''.j + 1 + 2k + k^* + l = W'.j$. Then, we know that

$$W'.\Xi_i \triangleright cod(\varepsilon_i)([((\varepsilon_i' \mathbin{⨾} dom(\varepsilon_i^\rho)) \mathbin{⨾} dom(\varepsilon_i))u_i' :: G_{1i})/x]t_i) :: \rho(G_2'') \longrightarrow^* W'''.\Xi_i \triangleright v_i'^*$$

We instantiate the induction hypothesis in the previous result $((W''', v'^*_1, v'^*_2) \in \mathcal{V}_\rho[\![G_2'']\!])$ with the type G_2' and the evidence $cod(\varepsilon) \vdash \Xi; \Delta \vdash G_2'' \sim G_2'$, where $cod(\varepsilon_1^\rho).n = l$, then we obtain that:

$$(W''', cod(\varepsilon_1^\rho)v'^*_1 :: \rho(G_2'), cod(\varepsilon_2^\rho)v'^*_2 :: \rho(G_2'))' \in \mathcal{T}_\rho[\![G_2']\!]$$

Then v'^*_i has to have the form: $v'^*_i = (\varepsilon_i'' \mathbin{⨾} cod(\varepsilon_i))u_{if} :: \rho(G_2'')$ form some ε_i'', u_{if}. Then as $(\varepsilon_1'' \mathbin{⨾} cod(\varepsilon_1)) \mathbin{⨾} cod(\varepsilon_1^\rho) = \varepsilon_1'' \mathbin{⨾} (cod(\varepsilon_1) \mathbin{⨾} cod(\varepsilon_1^\rho))$, then $(cod(\varepsilon_1) \mathbin{⨾} cod(\varepsilon_1^\rho))$ must behave uniformly (either the two of them fail, or the two of them does not fail). Thus, we get that $(\downarrow_l W''', v_{1f}, v_{2f}) \in \mathcal{V}_\rho[\![G_2']\!]$ where $v_{if} = (\varepsilon_i'' \mathbin{⨾} (cod(\varepsilon_i) \mathbin{⨾} cod(\varepsilon_i^\rho)))u_{if} :: \rho(G_2')$ and $W'''.j + 1 + 2k + 2l + k = W'.j$. Therefore, the result immediately.

Case (**Universal Type:** $G = \forall X.G_1''$ and $G' = \forall X.G_1'$). We know that:

$$(W, v_1, v_2) \in \mathcal{V}_\rho[\![\forall X.G_1'']\!]$$

Where $v_i = \varepsilon_i(\Lambda X.t_i) :: \forall X.\rho(G_1'')$ and $\varepsilon_i \vdash W.\Xi_i \vdash G_i \sim \forall X.\rho(G_1'')$.

We have to prove that:

$$(W, \varepsilon_1^\rho v_1 :: \forall X.\rho(G_1'), \varepsilon_2^\rho v_2 :: \forall X.\rho(G_1')) \in \mathcal{T}_\rho[\![\forall X.G_1']\!]$$

As $(\varepsilon_i \mathbin{⨾} \varepsilon_i^\rho)$ does not fail, then by the definition of $\mathcal{T}_\rho[\![\forall X.G_1']\!]$ we have to prove that:

$$(\downarrow_k W, (\varepsilon_1 \mathbin{⨾} \varepsilon_1^\rho)(\Lambda X.t_1) :: \forall X.\rho(G_1'), (\varepsilon_2 \mathbin{⨾} \varepsilon_2^\rho)(\Lambda X.t_2) :: \forall X.\rho(G_1')) \in \mathcal{V}_\rho[\![\forall X.G_1']\!]$$

or what is the same:

$$\begin{array}{l}
\forall W'' \geq (\downarrow_k W).\forall t_1', t_2', G_1^*, G_2^*, \alpha, \varepsilon_{11}, \varepsilon_{21}.\forall R \in \textsc{Rel}_{W''.j}[G_1^*, G_2^*].\\
(W''.\Xi_1 \vdash G_1^* \land W''.\Xi_2 \vdash G_2^* \land\\
W''.\Xi_1 \triangleright ((\varepsilon_1 \mathbin{⨾} \varepsilon_1^\rho)u_1 :: \forall X.G_1')[G_1^*] \longrightarrow W''.\Xi_1, \alpha := G_1^* \triangleright \varepsilon_{11}t_1' :: G_1'[G_1^*/X] \land\\
W''.\Xi_2 \triangleright ((\varepsilon_2 \mathbin{⨾} \varepsilon_2^\rho)u_2 :: \forall X.G_1')[G_2^*] \longrightarrow W''.\Xi_2, \alpha := G_2^* \triangleright \varepsilon_{21}t_2' :: G_1'[G_2^*/X]) \Rightarrow\\
(W''', t_1', t_2') \in \mathcal{T}_{\rho[X \mapsto \alpha]}[\![G_1']\!]
\end{array}$$

where $W''' = \downarrow(W'' \boxtimes (\alpha, G_1^*, G_2^*, R))$. Note that by the reduction rule of type application, we obtain that:

$$W''.\Xi_i \triangleright ((\varepsilon_i \mathbin{⨾} \varepsilon_i^\rho)\Lambda X.t_i :: \forall X.\rho(G_1'))\ [G_i^*] \longrightarrow$$
$$W''.\Xi_i, \alpha := G_i^* \triangleright \varepsilon_{\forall X.\rho(G_1')}^{E_i/\alpha^{E_i}}((\varepsilon_i \mathbin{⨾} \varepsilon_i^\rho)[\alpha^{E_i}]t_i[\alpha^{E_i}/X] :: \rho(G_1')[\alpha/X]) :: \rho(G_1')[G_i^*/X]$$

where $E_i = lift_{(W''.\Xi_i)}(G_i^*)$. The resulting evidences $\varepsilon_i \mathbin{;} \varepsilon_i^\rho$ have the form: $\langle \forall X.E_{i1}, \forall X.E_{i2}\rangle$, then:

$$\varepsilon_{\forall X.\rho(G_1')}^{E_i/\alpha^{E_i}}((\varepsilon_i \mathbin{;} \varepsilon_i^\rho)[\alpha^{E_i}]t_i[\alpha^{E_i}/X] :: \rho(G_1')[\alpha/X]) :: \rho(G_1')[G_i^*/X] =$$
$$\varepsilon_{\varepsilon_{\forall X.\rho(G_1')}}^{E_i/\alpha^{E_i}}(\langle E_{i1}[\alpha^{E_i}/X], E_{i2}[\alpha^{E_i}/X]\rangle t_i[\alpha^{E_i}/X] :: \rho(G_1')[\alpha/X])$$

Then we have to prove that:

$$(W''', (\langle E_{11}[\alpha^{E_1}/X], E_{12}[\alpha^{E_1}/X]\rangle t_1[\alpha^{E_1}/X] :: \rho(G_1')[\alpha/X]), (\langle E_{21}[\alpha^{E_2}/X], E_{22}[\alpha^{E_2}/X]\rangle t_2[\alpha^{E_2}/X] :: \rho(G_1')[\alpha/X]))$$
$$\in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G_1']\!]$$

Also by the Proposition 6.22 we know that:

$$(\varepsilon_i \mathbin{;} \varepsilon_i^\rho)[\alpha^{E_i}] = (\varepsilon_i[\alpha^{E_i}]) \mathbin{;} (\varepsilon_i^\rho[\alpha^{E_i}])$$

Note that:

$$(\varepsilon_i \mathbin{;} \varepsilon_i^\rho)[\alpha^{E_i}] = \langle E_{i1}[\alpha^{E_i}/X], E_{i2}[\alpha^{E_i}/X]\rangle = (\varepsilon_i[\alpha^{E_i}]) \mathbin{;} (\varepsilon_i^\rho[\alpha^{E_i}])$$

Then we have to prove that:

$$(W''', (\varepsilon_1[\alpha^{E_1}] \mathbin{;} \varepsilon_1^\rho[\alpha^{E_1}])t_1[\alpha^{E_1}/X] :: G_1'[\alpha/X]), (\varepsilon_2[\alpha^{E_2}] \mathbin{;} \varepsilon_2^\rho[\alpha^{E_2}])t_2[\alpha^{E_2}/X] :: \rho(G_1')[\alpha/X]))$$
$$\in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G_1']\!]$$

We know that

$$W'''.\Xi_1 \triangleright (\varepsilon_1[\alpha^{E_1}] \mathbin{;} \varepsilon_1^\rho[\alpha^{E_1}])t_1 :: G_1'[\alpha/X]) \longmapsto^{k^*}$$
$$\Xi_1 \triangleright (\varepsilon_1[\alpha^{E_1}] \mathbin{;} \varepsilon_1^\rho[\alpha^{E_1}])v_{1f} :: G_1'[\alpha/X]) \longmapsto^{k+l}$$
$$\Xi_1 \triangleright v_1^*$$

Note that by the reduction rule of type application, we obtain that:

$$W''.\Xi_i \triangleright (\varepsilon_i \Lambda X.t_i :: \forall X.\rho(G_1''))\ [G_i^*] \longrightarrow$$
$$W''.\Xi_i, \alpha := G_i^* \triangleright \varepsilon_{\forall X.\rho(G_1'')}^{E_i/\alpha^{E_i}}(\varepsilon_i[\alpha^{E_i}]t_i[\alpha^{E_i}/X] :: \rho(G_1'')[\alpha/X]) :: \rho(G_1'')[G_i^*/X]$$

Note that the evidence ε_i has the form: $\langle \forall X.E_{i1}'', \forall X.E_{i2}''\rangle$, then:

$$\varepsilon_{\forall X.\rho(G_1'')}^{E_i/\alpha^{E_i}}(\varepsilon_i[\alpha^{E_i}]t_i[\alpha^{E_i}/X] :: \rho(G_1'')[\alpha/X]) :: \rho(G_1'')[G_i^*/X] =$$
$$\varepsilon_{\varepsilon_{\forall X.\rho(G_1'')}}^{E_i/\alpha^{E_i}}(\langle E_{i1}''[\alpha^{E_i}/X], E_{i2}''[\alpha^{E_i}/X]\rangle t_i[\alpha^{E_i}/X] :: \rho(G_1'')[\alpha/X])$$

As we know that $(W, v_1, v_2) \in \mathcal{V}_\rho[\![\forall X.G_1'']\!]$, then we can instantiate with $\forall W'' \geq W$, G_1^*, G_2^*, R, $\varepsilon_1[\alpha^{E_1}]t_1[\alpha^{E_1}/X] :: \rho(G_1'')[\alpha/X]$, $\varepsilon_2[\alpha^{E_2}]t_2[\alpha^{E_2}/X] :: \rho(G_1'')[\alpha/X]$, $\varepsilon_{\varepsilon_{\forall X.\rho(G_1'')}}^{E_1/\alpha^{E_1}}$ and $\varepsilon_{\varepsilon_{\forall X.\rho(G_1'')}}^{E_2/\alpha^{E_2}}$.

Then we know that:

$$(W''', \varepsilon_1[\alpha^{E_1}]t_1[\alpha^{E_1}/X] :: \rho(G_1'')[\alpha/X]), \varepsilon_2[\alpha^{E_2}]t_2[\alpha^{E_2}/X] :: \rho(G_1'')[\alpha/X])) \in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G_1'']\!]$$

If the following term reduces to error, then the result follows immediately.

$$W'''.\Xi_1 \triangleright \varepsilon_1[\alpha^{E_1}]t_1[\alpha^{E_1}/X] :: \rho(G_1'')[\alpha/X])$$

If the above is not true, then the following terms reduce to values (v'_{if}) and $\exists W'''' \geq W'''$ such that $(W'''', v'_{1f}, v'_{2f}) \in \mathcal{V}_{\rho[X \mapsto \alpha]}[\![G''_1]\!]$ and $W''''.j + k^* + m = W'''.j$.

$$W'''.\Xi_i \triangleright \varepsilon_i[\alpha^{E_i}]t_i[\alpha^{E_i}/X] :: \rho(G''_1)[\alpha/X]) \longrightarrow^* W''''.\Xi_i \triangleright v'_{if}$$

Note that

$$\begin{aligned} W'''.\Xi_1 \triangleright \varepsilon_1[\alpha^{E_1}]t_1[\alpha^{E_1}/X] :: \rho(G''_1)[\alpha/X]) \longrightarrow^{k^*} \\ W''''.\Xi_1 \triangleright \varepsilon_1[\alpha^{E_1}]v_{1f} :: \rho(G''_1)[\alpha/X]) \longrightarrow^{m} \\ W''''.\Xi_i \triangleright v'_{1f} \end{aligned}$$

By definition of consistency and the evidence we know that $\varepsilon[X] \vdash W''''.\Xi; \Delta, X \vdash G''_1 \sim G'_1$. Then we instantiate the induction hypothesis in the previous result with $G = G'_1$ and $\varepsilon = \varepsilon[X]$. Calling $\rho' = \rho[X \mapsto \alpha]$, then we obtain that:

$$(W'''', \rho'_1(\varepsilon[X])v_{1f} :: \rho'(G'_1), \rho'_2(\varepsilon[X])v_{2f} :: \rho'(G'_1)) \in \mathcal{T}_{\rho'}[\![G'_1]\!]$$

but as $\rho'_1(\varepsilon[X]) = \varepsilon^{\rho}_i[\alpha^{E_i}]$ which is equivalent to

$$(W'''', (\varepsilon^{\rho}_1[\alpha^{E_1}])v_{1f} :: \rho(G'_1)[\alpha/X], (\varepsilon^{\rho}_2[\alpha^{E_2}])v_{2f} :: \rho(G'_1)[\alpha/X]) \in \mathcal{T}_{\rho'}[\![G'_1]\!]$$

Therefore,

$$(\downarrow_k W'''', v^*_1, v^*_2) \in \mathcal{T}_{\rho'}[\![G'_1]\!]$$

where $(\downarrow_k W'''').j + k^* + k + m = W'''.j$, and the result follows immediately.

Case (**Pairs:** $G = G_1 \times G2$). Similar to function case.

Case (A)(**Type Names:** $G = \alpha$). This means that $\alpha \in dom(\Xi)$. We know that $(W, \varepsilon_1 u_1 :: \alpha, \varepsilon_2 u_2 :: \alpha) \in \mathcal{V}_{\rho}[\![\alpha]\!]$ and $\varepsilon_i \vdash W.\Xi_i \vdash G_i \sim \alpha$, then $\varepsilon_i = \langle E_i, \alpha^{E'_i} \rangle$. Also we know that $\varepsilon \vdash \Xi; \Delta \vdash \alpha \sim G'$, therefore $\varepsilon = \langle \alpha^{E^*_1}, E^*_2 \rangle$, and $\varepsilon^{\rho}_i = \langle \alpha^{E^*_1}, E^*_2 \rangle = \varepsilon$, because ε can not have free type variable, so $\varepsilon \vdash \Xi \vdash \alpha \sim G'$. Since $(W, v_1, v_2) \in \mathcal{V}_{\rho}[\![\alpha]\!]$, we instantiate its definition with $\varepsilon \vdash \Xi \vdash \alpha \sim G'$, Ξ, such that $W \in \mathcal{S}[\![\Xi]\!]$ and G'. Therefore, we know that $(W, \varepsilon v_1 :: G', \varepsilon v_2 :: G')$, and the results follows immediately.

Case (B)(**Type Variables:** $G = X$). Suppose that $\rho(X) = \alpha$. We know that $\alpha \notin \Xi$, i.e. α may not be in sync, that $(W, \varepsilon_1 u_1 :: \alpha, \varepsilon_2 u_2 :: \alpha) \in \mathcal{V}_{\rho}[\![X]\!]$ and that $\varepsilon_i \vdash W.\Xi_i \vdash G_i \sim \alpha$, then $\varepsilon_i = \langle E_i, \alpha^{E'_i} \rangle$.

Then by construction of evidences, ε must be either $\langle X, X \rangle$ or $\langle ?, ? \rangle$ (any other case will fail when the meet is computed).

- ($\varepsilon = \langle X, X \rangle$). Then $\varepsilon^{\rho}_i = \langle \rho_i(X), \rho_i(X) \rangle$. But $\rho_i(X)$ is the type that contains the initial precision for α. Therefore $\alpha^{E'_i} \sqsubseteq \rho_i(X)$, and by Lemma 6.30, $\varepsilon_i \circ\!\!\!\!\!\;\, ; \varepsilon^{\rho}_i = \varepsilon_i$ and the result holds immediately by Lemma 6.14 (notice that if $G' = ?$ then we have to show that they are related to α which is part of the premise).
- ($\varepsilon = \langle ?, ? \rangle$). By Lemma 6.30 ($\varepsilon^{\rho}_i = \langle ?, ? \rangle$), $\varepsilon_i \circ\!\!\!\!\!\;\, ; \langle ?, ? \rangle = \varepsilon_i$ and the result holds immediately by Lemma 6.14.

Case (C)(**Unknown:** $G = ?$). We know that $(W, \varepsilon_1 u_1 :: ?, \varepsilon_2 u_2 :: ?) \in \mathcal{V}_{\rho}[\![?]\!]$ and $\varepsilon_i \vdash W.\Xi_i \vdash G_i \sim ?$. We are going to proceed by case analysis on ε_i:

(C.i) ($\varepsilon_i = \langle E_i, \alpha^{E'_i} \rangle$). Then this means we know that

$$(W, \varepsilon_1 u_1 :: \alpha, \varepsilon_2 u_2 :: \alpha) \in \mathcal{V}_{\rho}[\![\alpha]\!]$$

and $\varepsilon_i \vdash W.\Xi_i \vdash G_i \sim \alpha$, then $\varepsilon_i = \langle E_i, \alpha^{E'_i} \rangle$.

(a) (Case $\varepsilon = \langle \alpha^{E_3}, E_4\rangle$). Then as $\langle E_i, \alpha^{E'_i}\rangle \Vdash \Xi; \Delta \vdash G_i \sim ?$, then by Lemma 6.27 $\langle E_i, \alpha^{E'_i}\rangle \Vdash \Xi; \Delta \vdash G_i \sim \alpha$. Also we know that $? \sqsubseteq G$, then $G = ?$, and $\alpha \sqsubseteq G$. Finally, we reduce this case to the Case A if $\alpha \in \Xi$ or Case B if $\alpha \notin \Xi$.

(b) ($\varepsilon = \langle ?, ?\rangle$). Then $G' = ?$, and does $\varepsilon_i \,⨾\, \varepsilon = \varepsilon_i$. Then we have to prove that $(\downarrow_k W, \varepsilon_1 u_1 :: ?, \varepsilon_2 u_2 :: ?) \in \mathcal{V}_\rho[\![?]\!]$, and as $const(\alpha^{E'_i}) = \alpha$ that is equivalent to prove that $(\downarrow_k W, \varepsilon_1 u_1 :: \alpha, \varepsilon_2 u_2 :: \alpha) \in \mathcal{V}_\rho[\![\alpha]\!]$ which follows by the premise and Lemma 6.14.

(c) ($\varepsilon = \langle ?, \beta^{\beta'^{\dots ?}}\rangle$). Where β cannot transitively point to some unsync variable. Then by definition of the transitivity operator, $\varepsilon_i \,⨾\, \varepsilon = \langle E''_i, \beta^{E'''_i}\rangle$ (where $\langle E_i, \alpha^{E'_i}\rangle \,⨾\, \langle ?, \beta'^{\dots ?}\rangle = \langle E''_i, E'''_i\rangle$). Then we have to prove that

$$(\downarrow_k W, \langle E''_1, \beta^{E'''_1}\rangle u_1 :: G', \langle E''_2, \beta^{E'''_2}\rangle u_2 :: G') \in \mathcal{V}_\rho[\![G']\!]$$

where G' is either ? or β. In any case this is equivalent to prove that

$$(\downarrow_k W, \langle E''_1, \beta^{E'''_1}\rangle u_1 :: \beta, \langle E''_2, \beta^{E'''_2}\rangle u_2 :: \beta) \in \mathcal{V}_\rho[\![\beta]\!]$$

Therefore, we have to prove

$$(\downarrow_{k-1} W, \langle E''_1, E'''_1\rangle u_1 :: G'', \langle E''_2, E'''_2\rangle u_2 :: G'') \in \mathcal{V}_\rho[\![G'']\!]$$

where $G'' = W.\Xi_1(\beta) = W.\Xi_2(\beta)$ (note that β is sync). As $\langle E_i, \alpha^{E'_i}\rangle \,⨾\, \langle ?, \beta'^{\dots ?}\rangle = \langle E''_i, E'''_i\rangle$, then we can reduce the demonstration to prove that:

$$(\downarrow_{k-1} W, (\langle E_1, \alpha^{E'_1}\rangle \,⨾\, \langle ?, \beta'^{\dots ?}\rangle) u_1 :: G'', (\langle E_2, \alpha^{E'_2}\rangle \,⨾\, \langle ?, \beta'^{\dots ?}\rangle) u_2 :: G'') \in \mathcal{V}_\rho[\![G'']\!]$$

Thus, we reduce this case to this same case (note that we have base case because the sequence ends in ?).

Also, we have to prove that ($\forall \Xi', \varepsilon', G^*$ such that ($\downarrow_{k-1} W \in \mathcal{S}[\![\Xi']\!] \wedge \varepsilon' \vdash \Xi' \vdash \beta \sim G^*$), we get that

$$(\downarrow_{k-1} W, \varepsilon'(\langle E''_1, \beta^{E'''_1}\rangle u_1 :: \beta) :: G^*, \varepsilon'(\langle E''_2, \beta^{E'''_2}\rangle u_2 :: \beta) :: G^*) \in \mathcal{T}_\rho[\![G^*]\!])$$

or what is the same (($\langle E''_1, \beta^{E'''_1}\rangle \,⨾\, \varepsilon'$) fails the result follows immediately)

$$(\downarrow_{k-1-k'} W, (\langle E''_1, \beta^{E'''_1}\rangle \,⨾\, \varepsilon') u_1 :: G^*, (\langle E''_2, \beta^{E'''_2}\rangle \,⨾\, \varepsilon') u_2 :: G^*) \in \mathcal{V}_\rho[\![G^*]\!])$$

where $\varepsilon' = \langle \beta^{E^*_1}, E^*_2\rangle$, $\varepsilon'.n = k'$ and $G'' = W'.\Xi_1(\beta) = W'.\Xi_2(\beta)$. By definition of transitivity and Lemma 6.30, we know that

$$\langle E''_i, \beta^{E'''_i}\rangle \,⨾\, \langle \beta^{E^*_1}, E^*_2\rangle = \langle E''_i, E'''_i\rangle \,⨾\, \langle E^*_1, E^*_2\rangle$$

$$\langle E_i, \alpha^{E'_i}\rangle \,⨾\, \langle ?, \beta'^{\dots ?}\rangle = \langle E^*_{1i}, \beta'^{E^*_{2i}}\rangle = \langle E''_i, E'''_i\rangle$$

Thus $G'' = \beta'$ or $G'' = ?$, in any case we know that $(\downarrow_{k-1} W, \langle E''_1, E'''_1\rangle u_1 :: \beta', \langle E''_2, E'''_2\rangle u_2 :: \beta') \in \mathcal{V}_\rho[\![\beta']\!]$.

We know that $\langle E^*_1, E^*_2\rangle \vdash \Xi' \vdash G'' \sim G^*$. Since $\langle E^*_1, E^*_2\rangle \vdash \Xi \vdash G'' \sim G^*$, $\downarrow_{k-1} W \in \mathcal{S}[\![\Xi']\!]$, $(\downarrow_{k-1} W, \langle E''_1, E'''_1\rangle u_1 :: \beta', \langle E''_2, E'''_2\rangle u_2 :: \beta') \in \mathcal{V}_\rho[\![\beta']\!]$, by the definition of $\mathcal{V}_\rho[\![\beta']\!]$, we know that (since ($\langle E''_1, E'''_1\rangle \,⨾\, \varepsilon'$) does not fail then ($\langle E''_1, E'''_1\rangle \,⨾\, \langle E^*_1, E^*_2\rangle$) also does not fail by the transitivity rules and $\langle E^*_1, E^*_2\rangle \vdash \Xi' \vdash \beta' \sim G^*$)

$$(\downarrow_{k-1-k'} W, (\langle E''_1, E'''_1\rangle \,⨾\, \langle E^*_1, E^*_2\rangle) u_1 :: G^*, (\langle E''_2, E'''_2\rangle \,⨾\, \langle E^*_1, E^*_2\rangle) u_2 :: G^*) \in \mathcal{V}_\rho[\![G^*]\!])$$

The result follows immediately.

(d) ($\varepsilon = \langle ?, \beta^{?}\rangle$). Then by definition of the transitivity operator, $\varepsilon_i \mathbin{⨾} \varepsilon = \langle E_i, \beta^{\alpha^{E'_i}}\rangle$. Then we have to prove that

$$(\downarrow_k W, \langle E_1, \beta^{\alpha^{E'_1}}\rangle u_1 :: G', \langle E_2, \beta^{\alpha^{E'_2}}\rangle u_2 :: G') \in \mathcal{V}_\rho[\![G']\!]$$

where G' is either ? or β. In any case this is equivalent to prove that
$(\downarrow_k W, \langle E_1, \beta^{\alpha^{E'_1}}\rangle u_1 :: \beta, \langle E_2, \beta^{\alpha^{E'_2}}\rangle u_2 :: \beta) \in \mathcal{V}_\rho[\![\beta]\!]$
Therefore, we have to prove that
$(\downarrow_{k-1} W, \langle E_1, \alpha^{E'_1}\rangle u_1 :: G'', \langle E_2, \alpha^{E'_2}\rangle u_2 :: G'') \in \mathcal{V}_\rho[\![G'']\!]$ where $G'' = W.\Xi_1(\beta) = W.\Xi_2(\beta) =$? (note that β is sync). Therefore, we have to prove that $(\downarrow_{k-1} W, \langle E_1, \alpha^{E'_1}\rangle u_1 :: \alpha, \langle E_2, \alpha^{E'_2}\rangle u_2 :: \alpha) \in \mathcal{V}_\rho[\![\alpha]\!]$ which follows immediately by premise and Lemma 6.14.
Also, we have to prove that ($\forall \Xi', \varepsilon', G^*$ such that $(\downarrow_{k-1} W \in \mathcal{S}[\![\Xi']\!] \wedge \varepsilon' \vdash \Xi' \vdash \beta \sim G^*)$, we get that

$$(\downarrow_{k-1} W, \varepsilon'(\langle E_1, \beta^{\alpha^{E'_1}}\rangle u_1 :: \beta) :: G^*, \varepsilon'(\langle E_2, \beta^{\alpha^{E'_2}}\rangle u_2 :: \beta) :: G^*) \in \mathcal{T}_\rho[\![G^*]\!])$$

or what is the same (($\langle E_1, \beta^{\alpha^{E'_1}}\rangle \mathbin{⨾} \varepsilon'$) fails the result follows immediately)

$$(\downarrow_{k-1-k'} W, (\langle E_1, \beta^{\alpha^{E'_1}}\rangle \mathbin{⨾} \varepsilon')u_1 :: G^*, (\langle E_2, \beta^{\alpha^{E'_2}}\rangle \mathbin{⨾} \varepsilon')u_2 :: G^*) \in \mathcal{V}_\rho[\![G^*]\!])$$

where $\varepsilon' = \langle \beta^{E^*_1}, E^*_2\rangle$, $\varepsilon'.n = k'$ and $G'' = W'.\Xi_1(\beta) = W'.\Xi_2(\beta) =$?. By definition of transitivity and Lemma 6.30, we know that

$$\langle E_i, \beta^{\alpha^{E'_i}}\rangle \mathbin{⨾} \langle \beta^{E^*_1}, E^*_2\rangle = \langle E_i, \alpha^{E'_i}\rangle \mathbin{⨾} \langle E^*_1, E^*_2\rangle$$

We know that $\langle E^*_1, E^*_2\rangle \vdash \Xi' \vdash G'' \sim G^*$. Since $\langle E^*_1, E^*_2\rangle \vdash \Xi \vdash G'' \sim G^*$, $\downarrow_{k-1} W \in \mathcal{S}[\![\Xi']\!]$, $(\downarrow_{k-1} W, \langle E_1, \alpha^{E'_1}\rangle u_1 :: \alpha, \langle E_2, \alpha^{E'_2}\rangle u_2 :: \alpha) \in \mathcal{V}_\rho[\![\alpha]\!]$, by the definition of $\mathcal{S}[\![\Xi]\!]\alpha$, we know that (since ($\langle E_1, \alpha^{E'_1}\rangle \mathbin{⨾} \varepsilon'$) does not fail then ($\langle E_1, \alpha^{E'_1}\rangle \mathbin{⨾} \langle E^*_1, E^*_2\rangle$) also does not fail by the transitivity rules and $\langle E^*_1, E^*_2\rangle \vdash \Xi' \vdash \alpha \sim G^*$)

$$(\downarrow_{k-1-k'} W, (\langle E_1, \alpha^{E'_1}\rangle \mathbin{⨾} \langle E^*_1, E^*_2\rangle)u_1 :: G^*, (\langle E_2, \alpha^{E'_2}\rangle \mathbin{⨾} \langle E^*_1, E^*_2\rangle)u_2 :: G^*) \in \mathcal{V}_\rho[\![G^*]\!])$$

The result follows immediately.

(C.ii) ($\varepsilon_i = \langle H_{i1}, H_{i2}\rangle$). Let $G'' = const(H_{i2})$, and we know that $G'' \in$ HeadType. By unfolding of the logical relation for ?, we also know that

$$(W, \langle H_{11}, H_{12}\rangle u_1 :: G'', \langle H_{21}, H_{22}\rangle u_2 :: G'') \in \mathcal{V}_\rho[\![G'']\!]$$

and we have to prove that

$$(\downarrow_k W, (\langle H_{11}, H_{12}\rangle \mathbin{⨾} \varepsilon^\rho_1)u_1 :: G', (\langle H_{21}, H_{22}\rangle \mathbin{⨾} \varepsilon^\rho_2)u_2 :: G') \in \mathcal{V}_\rho[\![G']\!]$$

Note that for consistent transitivity to hold, then ε has to take the following forms:

(a) $\varepsilon = \langle H_3, E_4\rangle$. Then as $\varepsilon \Vdash \Xi; \Delta \vdash ? \sim G'$, by Lemma 6.27, $\varepsilon \Vdash \Xi; \Delta \vdash const(H_3) \sim G'$, and we proceed just like Case D, where $G \in$ HeadType ($G = G''$).

(b) $\varepsilon = \langle ?, ?\rangle$. Then $G' =$? and $\langle H_{i1}, H_{i2}\rangle \mathbin{⨾} \langle ?, ?\rangle = \langle H_{i1}, H_{i2}\rangle$. The result follows immediately by premise and Lemma 6.14.

(c) $\varepsilon = \langle ?, \alpha^{?}\rangle$. Then we know that $W.\Xi_i(\alpha) =$?, and by inspection of the consistent transitivity rules, $\langle H_{i1}, H_{i2}\rangle \mathbin{⨾} \langle ?, \alpha^{?}\rangle = \langle H_{i1}, \alpha^{H_{i2}}\rangle$. Then by definition of the interpretation of G', which may be ? or α), in any case, we have to prove that
$(\downarrow_k W, \langle H_{11}, \alpha^{H_{12}}\rangle u_1 :: \alpha, \langle H_{21}, \alpha^{H_{22}}\rangle u_2 :: \alpha) \in \mathcal{V}_\rho[\![\alpha]\!]$
Therefore, we have to prove that $(\downarrow_{k-1} W, \langle H_{11}, H_{12}\rangle u_1 :: ?, \langle H_{21}, H_{22}\rangle u_2 :: ?) \in \mathcal{V}_\rho[\![?]\!]$ which follows by premise and Lemma 6.14.

Also, we have to prove that ($\forall \Xi', \varepsilon', G^*$ such that $(\downarrow_{k-1} W \in \mathcal{S}[\![\Xi']\!] \wedge \varepsilon' \vdash \Xi' \vdash \beta \sim G^*)$, we get that

$$(\downarrow_{k-1} W, \varepsilon'(\langle H_{11}, \alpha^{H_{12}} \rangle u_1 :: \alpha) :: G^*, \varepsilon'(\langle H_{21}, \alpha^{H_{22}} \rangle u_2 :: \alpha) :: G^*) \in \mathcal{T}_\rho[\![G^*]\!])$$

or what is the same (($\langle H_{11}, \alpha^{H_{12}} \rangle \,\mathring{,}\, \varepsilon'$) fails the result follows immediately)

$$(\downarrow_{k-1-k'} W, (\langle H_{11}, \alpha^{H_{12}} \rangle \,\mathring{,}\, \varepsilon') u_1 :: G^*, (\langle H_{21}, \alpha^{H_{22}} \rangle \,\mathring{,}\, \varepsilon') u_2 :: G^*) \in \mathcal{V}_\rho[\![G^*]\!])$$

where $\varepsilon' = \langle \alpha^{H_1^*}, E_2^* \rangle$, $\varepsilon'.n = k'$. By definition of transitivity and Lemma 6.30, we know that

$$\langle H_{i1}, \alpha^{H_{i2}} \rangle \,\mathring{,}\, \langle \alpha^{H_1^*}, E_2^* \rangle = \langle H_{i1}, H_{i2} \rangle \,\mathring{,}\, \langle H_1^*, E_2^* \rangle$$

Therefore, we have to prove that

$$(\downarrow_{k-1-k'} W, (\langle H_{11}, H_{12} \rangle \,\mathring{,}\, \langle H_1^*, E_2^* \rangle) u_1 :: G^*, (\langle H_{21}, H_{22} \rangle \,\mathring{,}\, \langle H_1^*, E_2^* \rangle) u_2 :: G^*) \in \mathcal{V}_\rho[\![G^*]\!])$$

We know that $\langle E_1^*, E_2^* \rangle \vdash \Xi' \vdash ? \sim G^*$. Since $\langle E_1^*, E_2^* \rangle \vdash \Xi \vdash ? \sim G^*$, $\downarrow_{k-1} W \in \mathcal{S}[\![\Xi']\!]$, we follow by this Case(a), but with evidence $\langle H_1^*, E_2^* \rangle$. The result follows immediately.

(d) $\varepsilon = \langle ?, \alpha^{\beta^{E_4}} \rangle$. Then we know that $W.\Xi_i(\alpha) \in \{\beta, ?\}$ ($W.\Xi_i(\alpha) = G_{123}$) and by inspection of the consistent transitivity rules, $\langle H_{i1}, H_{i2} \rangle \,\mathring{,}\, \langle ?, \alpha^{\beta^{E_{i4}}} \rangle = \langle H'_{i1}, \alpha^{\beta^{E'_{i4}}} \rangle$, where $\langle H_{i1}, H_{i2} \rangle \,\mathring{,}\, \langle ?, E_{i4} \rangle = \langle H_{i1}, E'_{i4} \rangle$.
Then by definition of the interpretation of α (after one or two unfolding of $G' = ?$), we have to prove that
$(\downarrow_{k-1} W, (\langle H'_{11}, \beta^{E'_{14}} \rangle u_1 :: G_{123}), (\langle H'_{21}, \beta^{E'_{24}} \rangle u_2 :: G_{123})) \in \mathcal{T}_\rho[\![G_{123}]\!])$
or what is the same

$$\begin{aligned}&(\downarrow_{k-1} W, (\langle H_{11}, H_{12} \rangle \,\mathring{,}\, \langle ?, \beta^{E_{14}} \rangle) u_1 :: \beta,\\ &(\langle H_{21}, H_{22} \rangle \,\mathring{,}\, \langle ?, \beta^{E_{24}} \rangle) u_2 :: \beta) \in \mathcal{V}_\rho[\![\beta]\!]\end{aligned}$$

and then we proceed to the same case one more time (notice that the recursion is finite, until we get to the previous sub case).
Also, we have to prove that ($\forall \Xi', \varepsilon', G^*$ such that $(\downarrow_{k-1} W \in \mathcal{S}[\![\Xi']\!] \wedge \varepsilon' \vdash \Xi' \vdash \alpha \sim G^*)$, we get that

$$(\downarrow_{k-1} W, \varepsilon'(\langle H'_{11}, \alpha^{\beta^{E'_{14}}} \rangle u_1 :: \alpha) :: G^*, \varepsilon'(\langle H'_{21}, \alpha^{\beta^{E'_{24}}} \rangle u_2 :: \alpha) :: G^*) \in \mathcal{T}_\rho[\![G^*]\!])$$

or what is the same (($\langle H'_{11}, \alpha^{E'_{14}} \rangle \,\mathring{,}\, \varepsilon'$) fails the result follows immediately)

$$(\downarrow_{k-1-k'} W, (\langle H'_{11}, \alpha^{\beta^{E'_{14}}} \rangle \,\mathring{,}\, \varepsilon') u_1 :: G^*, (\langle H'_{21}, \alpha^{\beta^{E'_{24}}} \rangle \,\mathring{,}\, \varepsilon') u_2 :: G^*) \in \mathcal{V}_\rho[\![G^*]\!])$$

where $\varepsilon' = \langle \alpha^{E_1^*}, E_2^* \rangle$, $\varepsilon'.n = k'$. By definition of transitivity and Lemma 6.30, we know that

$$\langle H'_{i1}, \alpha^{\beta^{E'_{i4}}} \rangle \,\mathring{,}\, \langle \alpha^{E_1^*}, E_2^* \rangle = \langle H'_{i1}, \beta^{E'_{i2}} \rangle \,\mathring{,}\, \langle E_1^*, E_2^* \rangle$$

Therefore, we have to prove that

$$(\downarrow_{k-1-k'} W, (\langle H'_{11}, \beta^{E'_{14}} \rangle \,\mathring{,}\, \langle E_1^*, E_2^* \rangle) u_1 :: G^*, (\langle H_{21}, \beta^{E_{24}} \rangle \,\mathring{,}\, \langle E_1^*, E_2^* \rangle) u_2 :: G^*) \in \mathcal{V}_\rho[\![G^*]\!])$$

We know that $\langle E_1^*, E_2^* \rangle \vdash \Xi' \vdash G_{123} \sim G^*$. Since $\langle E_1^*, E_2^* \rangle \vdash \Xi \vdash G_{123} \sim G^*$, $\downarrow_{k-1} W \in \mathcal{S}[\![\Xi']\!]$, and $(\downarrow_{k-1} W, (\langle H'_{11}, \beta^{E'_{14}} \rangle u_1 :: G_{123}), (\langle H'_{21}, \beta^{E'_{24}} \rangle u_2 :: G_{123})) \in \mathcal{T}_\rho[\![G_{123}]\!])$, by instantiating the definition of $\mathcal{V}_\rho[\![\beta]\!]$, the result follows immediately.

Case (D) (**Head Types:** $G \in$ HeadType). We know that $(W, \varepsilon_1 u_1 :: \rho(G), \varepsilon_2 u_2 :: \rho(G)) \in \mathcal{V}_\rho[\![G]\!]$ and $\varepsilon_i \vdash W.\Xi_i \vdash G_i \sim G$. Also $\varepsilon_i = \langle H_{i1}, H_{i2} \rangle$, for some H_{i1}, H_{i2}. We proceed by case analysis on G' and ε.

(D.i) ($\varepsilon = \langle H_3, \alpha^{E_4}\rangle$). Then $G' = \alpha$, or $G' = ?$. Notice that as α^{E_4} cannot have free type variables therefore H_3 neither. Then $\varepsilon = \rho_i(\varepsilon)$. As α is sync, then let us call $G'' = W.\Xi_i(\alpha)$. In either case $G' = \alpha$, or $G' = ?$, what we have to prove boils down to

$$(\downarrow_k W, (\varepsilon_1 \mathbin{⨾} \langle H_3, \alpha^{E_4}\rangle)u_1 :: \alpha, (\varepsilon_2 \mathbin{⨾} \langle H_3, \alpha^{E_4}\rangle)u_2 :: \alpha) \in \mathcal{V}_\rho[\![\alpha]\!]$$

Therefore, we have to prove that

$$(\downarrow_{k-1} W, (\varepsilon_1 \mathbin{⨾} \langle H_3, E_4\rangle)u_1 :: G'', (\varepsilon_2 \mathbin{⨾} \langle H_3, E_4\rangle)u_2 :: G'') \in \mathcal{V}_\rho[\![G'']\!]$$

Then we proceed by case analysis on ε:

- (Case $\varepsilon = \langle H_3, \alpha^{\beta^{E_4}}\rangle$). We know that $\alpha \sqsubseteq G'$ and that $\langle H_3, \alpha^{\beta^{E_4}}\rangle \Vdash \Xi; \Delta \vdash G \sim G'$, then by Lemma 6.27, we know that $\langle H_3, \alpha^{\beta^{E_4}}\rangle \vdash \Xi; \Delta \vdash G \sim \alpha$. Also by Lemma 6.29, $\langle H_3, \beta^{E_4}\rangle \vdash \Xi; \Delta \vdash G \sim G''$. As $\beta^{E_4} \sqsubseteq G''$, then G'' can either be ? or β.
  If $G'' = ?$, then by definition of $\mathcal{V}_\rho[\![?]\!]$, we have to prove that the resulting values belong to $\mathcal{V}_\rho[\![\beta]\!]$. Also as $\langle H_3, \beta^{E_4}\rangle \vdash \Xi; \Delta \vdash G \sim ?$, by Lemma 6.27, $\langle H_3, \beta^{E_4}\rangle \vdash \Xi; \Delta \vdash G \sim \beta$, and then we proceed just like this case once again (this is process is finite as there are no circular references by construction and it ends up in something different to a type name).
  If $G'' = \beta$ we use an analogous argument as for $G'' = ?$.
- (Case $\varepsilon = \langle H_3, \alpha^{H_4}\rangle$). Then we have to prove that

$$(\downarrow_{k-1} W, (\varepsilon_1 \mathbin{⨾} \langle H_3, H_4\rangle)u_1 :: G'', (\varepsilon_2 \mathbin{⨾} \langle H_3, H_4\rangle)u_2 :: G'') \in \mathcal{V}_\rho[\![G'']\!]$$

  By Lemma 6.29, $\langle H_3, H_4\rangle \vdash \Xi; \Delta \vdash G \sim G''$. Then if $G'' = ?$, we proceed as the case $G \in$ HeadType, $G' = ?$ with $\varepsilon = \langle H_3, H_4\rangle$ (Case (D.ii)). If $G'' \in$ HeadType, we proceed as the case $G \in$ HeadType, $G' \in$ HeadType with $\varepsilon = \langle H_3, H_4\rangle$, where $H_3, H_4 \in$ HeadType (Case (D.iii)).

Also, we have to prove that ($\forall \Xi', \varepsilon', G^*$ such that $(\downarrow_k W \in \mathcal{S}[\![\Xi']\!] \wedge \varepsilon' \vdash \Xi' \vdash \alpha \sim G^*) \wedge \varepsilon' = \langle \alpha^{E_5}, E_6\rangle \wedge \varepsilon'.n = k'$, we get that

$$(\downarrow_k W, \varepsilon'((\varepsilon_1 \mathbin{⨾} \langle H_3, \alpha^{H_4}\rangle)u_1 :: \alpha) :: G^*, \varepsilon'((\varepsilon_2 \mathbin{⨾} \langle H_3, \alpha^{H_4}\rangle)u_2 :: \alpha) :: G^*) \in \mathcal{T}_\rho[\![G^*]\!])$$

or what is the same $(((\varepsilon_1 \mathbin{⨾} \langle H_3, H_4\rangle) \mathbin{⨾} \langle E_5, E_6\rangle)$ fails the result follows immediately)

$$(\downarrow_{k-k'} W, (\varepsilon_1 \mathbin{⨾} (\langle H_3, H_4\rangle \mathbin{⨾} \langle E_5, E_6\rangle))u_1 :: G^*, (\varepsilon_2 \mathbin{⨾} (\langle H_3, H_4\rangle \mathbin{⨾} \langle E_5, E_6\rangle))u_2 :: G^*) \in \mathcal{V}_\rho[\![G^*]\!])$$

where $(\langle H_3, H_4\rangle \mathbin{⨾} \langle E_5, E_6\rangle).n = (k + k')$ We know that $(W, \varepsilon_1 u_1 :: \rho(G), \varepsilon_2 u_2 :: \rho(G)) \in \mathcal{V}_\rho[\![G]\!]$, therefore $(\downarrow_k W, \varepsilon_1 u_1 :: \rho(G), \varepsilon_2 u_2 :: \rho(G)) \in \mathcal{V}_\rho[\![G]\!]$, by Lemma 6.14, where now $\varepsilon_1.n = l + k$. Then we apply the induction hypothesis on $(\downarrow_k W, \varepsilon_1 u_1 :: \rho(G), \varepsilon_2 u_2 :: \rho(G)) \in \mathcal{V}_\rho[\![G]\!]$ and the evidence $(\langle H_3, H_4\rangle \mathbin{⨾} \langle G_5, G_6\rangle)$, but where $(\langle H_3, H_4\rangle \mathbin{⨾} \langle G_5, G_6\rangle).n = k'$. Therefore the results follows immediately:

$$(\downarrow_{k-k'} W, (\varepsilon_1 \mathbin{⨾} (\langle H_3, H_4\rangle \mathbin{⨾} \langle G_5, G_6\rangle))u_1 :: G^*, (\varepsilon_2 \mathbin{⨾} (\langle H_3, H_4\rangle \mathbin{⨾} \langle G_5, G_6\rangle))u_2 :: G^*) \in \mathcal{V}_\rho[\![G^*]\!])$$

(D.ii) ($G' = ?, \varepsilon = \langle H_3, H_4\rangle$). We have to prove that

$$(\downarrow_k W, (\varepsilon_1 \mathbin{⨾} \rho_1(\varepsilon))u_1 :: ?, (\varepsilon_2 \mathbin{⨾} \rho_2(\varepsilon))u_2 :: ?) \in \mathcal{V}_\rho[\![?]\!]$$

which is equivalent to prove that

$$(\downarrow_k W, (\varepsilon_1 \mathbin{⨾} \rho_1(\varepsilon))u_1 :: H, (\varepsilon_2 \mathbin{⨾} \rho_2(\varepsilon))u_2 :: H) \in \mathcal{V}_\rho[\![H]\!]$$

for $H = const(H_{i2})$ (and $H \in$ HeadType). But notice that as $\varepsilon \vdash \Xi; \Delta \vdash G \sim ?$, then as $H_4 \sqsubseteq H \sqsubseteq ?$, then by Lemma 6.27, $\varepsilon \vdash \Xi; \Delta \vdash G \sim H$, then we proceed just like the case $G \in$ HeadType and $G' \in$ HeadType (Case (D.iii)).

(D.iii) ($G' \in$ HeadType). These cases are already analyzed, by structural analysis of types (Case $G = G_1'' \rightarrow G_2''$ and $G' = G_1' \rightarrow G_2'$), (Case $G = \forall X.G_1''$ and $G' = \forall X.G_1'$), (Case $G = \langle G_1'', G_2'' \rangle$ and $G' = \langle G_1', G_2' \rangle$) and (Case $G = B$ and $G' = B$).

□

Lemma 10.5 (Ascriptions Preserve Relations). *If* $(W, v_1, v_2) \in \mathcal{V}_\rho[\![G]\!]$, $\varepsilon \Vdash \Xi; \Delta \vdash G \sim G'$, $W \in \mathcal{S}[\![\Xi]\!]$, *and* $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$, *then* $(W, \rho_1(\varepsilon)v_1 :: \rho(G'), \rho_2(\varepsilon)v_2 :: \rho(G')) \in \mathcal{T}_\rho[\![G']\!]$.

Proof. Direct by Prop. 6.17. □

Lemma 6.18 (Associativity of the evidence).

$$(\varepsilon_1 \mathbin{⨾} \varepsilon_2) \mathbin{⨾} \varepsilon_3 = \varepsilon_1 \mathbin{⨾} (\varepsilon_2 \mathbin{⨾} \varepsilon_3)$$

Proof. By induction on the structure of evidences.

Case ($\varepsilon_1 = \langle E_{11}, \alpha^{E_{12}} \rangle, \varepsilon_2 = \langle \alpha^{E_{21}}, E_{22} \rangle, \varepsilon_3 = \langle E_{31}, E_{32} \rangle$). By definition of consistent transitivity, we know that

- $(\varepsilon_1 \mathbin{⨾} \varepsilon_2) \mathbin{⨾} \varepsilon_3 = (\langle E_{11}, E_{12} \rangle \mathbin{⨾} \langle E_{21}, E_{22} \rangle) \mathbin{⨾} \langle E_{31}, E_{32} \rangle$
- $\varepsilon_1 \mathbin{⨾} (\varepsilon_2 \mathbin{⨾} \varepsilon_3) = \langle E_{11}, E_{12} \rangle \mathbin{⨾} (\langle E_{21}, E_{22} \rangle \mathbin{⨾} \langle E_{31}, E_{32} \rangle)$

Then by the induction hypothesis $(\langle E_{11}, E_{12} \rangle \mathbin{⨾} \langle E_{21}, E_{22} \rangle) \mathbin{⨾} \langle E_{31}, E_{32} \rangle = \langle E_{11}, E_{12} \rangle \mathbin{⨾} (\langle E_{21}, E_{22} \rangle \mathbin{⨾} \langle E_{31}, E_{32} \rangle)$, and the result follows immediately.

Case ($\varepsilon_1 = \langle E_{11}, E_{12} \rangle, \varepsilon_2 = \langle E_{21}, \alpha^{E_{22}} \rangle, \varepsilon_3 = \langle \alpha^{E_{31}}, E_{32} \rangle$). Similar to the previous.

Case ($\varepsilon_1 = \langle \alpha^{E_{11}}, E_{12} \rangle, \varepsilon_2 = \langle E_{21}, E_{22} \rangle, \varepsilon_3 = \langle E_{31}, E_{32} \rangle$). By definition of consistent transitivity, we know that

- $(\varepsilon_1 \mathbin{⨾} \varepsilon_2) \mathbin{⨾} \varepsilon_3 = \langle \alpha^{E_1}, E_2 \rangle \mathbin{⨾} \langle E_{31}, E_{32} \rangle = \langle \alpha^{E_1'}, E_2' \rangle$, where $\langle E_1, E_2 \rangle = (\langle E_{11}, E_{12} \rangle \mathbin{⨾} \langle E_{21}, E_{22} \rangle)$, $\langle E_1', E_2' \rangle = (\langle E_{11}, E_{12} \rangle \mathbin{⨾} \langle E_{21}, E_{22} \rangle) \mathbin{⨾} \langle E_{31}, E_{32} \rangle$.
- $\varepsilon_1 \mathbin{⨾} (\varepsilon_2 \mathbin{⨾} \varepsilon_3) = \langle \alpha^{E_{11}}, E_{12} \rangle \mathbin{⨾} (\langle E_{21}, E_{22} \rangle \mathbin{⨾} \langle E_{31}, E_{32} \rangle)$
- Note that by the induction hypothesis $\langle E_1', E_2' \rangle = (\langle E_{11}, E_{12} \rangle \mathbin{⨾} \langle E_{21}, E_{22} \rangle) \mathbin{⨾} \langle E_{31}, E_{32} \rangle = \langle E_{11}, E_{12} \rangle \mathbin{⨾} (\langle E_{21}, E_{22} \rangle \mathbin{⨾} \langle E_{31}, E_{32} \rangle)$

Then, the result follows immediately because $\langle \alpha^{E_{11}}, E_{12} \rangle \mathbin{⨾} (\langle E_{21}, E_{22} \rangle \mathbin{⨾} \langle E_{31}, E_{32} \rangle) = \langle \alpha^{E_1'}, E_2' \rangle$.

Case ($\varepsilon_1 = \langle E_{11}, E_{12} \rangle, \varepsilon_2 = \langle E_{21}, E_{22} \rangle, \varepsilon_3 = \langle E_{31}, \alpha^{E_{32}} \rangle$). Similar to the previous.

Case ($\varepsilon_1 = \langle ?, ? \rangle, \varepsilon_2 = \langle E_{21}, E_{22} \rangle, \varepsilon_3 = \langle E_{31}, E_{32} \rangle$). Trivially, by definition of consistent transitivity.

Case ($\varepsilon_1 = \langle E_{11}, E_{12} \rangle, \varepsilon_2 = \langle ?, ? \rangle, \varepsilon_3 = \langle E_{31}, E_{32} \rangle$). Trivially, by definition of consistent transitivity.

Case ($\varepsilon_1 = \langle E_{11}, E_{12} \rangle, \varepsilon_2 = \langle E_{21}, E_{22} \rangle, \varepsilon_3 = \langle ?, ? \rangle$). Trivially, by definition of consistent transitivity.

Case ($\varepsilon_1 = \langle E_{11}, E_{12} \rangle, \varepsilon_2 = \langle E_{21}, E_{22} \rangle, \varepsilon_3 = \langle ?, ? \rangle$). Trivially, by definition of consistent transitivity.

The other cases are pretty similar.

□

Lemma 6.19. *If* $(W, t_1, t_2) \in \mathcal{T}_\rho[\![G]\!]$, *then* $(\downarrow W, t_1, t_2) \in \mathcal{T}_\rho[\![G]\!]$

Proof. By definition of $\mathcal{T}_\rho[\![G]\!]$. □

Proposition 6.20. $dom(\varepsilon_1 \mathbin{⨾} \varepsilon_2) = dom(\varepsilon_2) \mathbin{⨾} dom(\varepsilon_1)$

Proof. Direct by inspection on the inductive definition of consistent transitivity. □

Proposition 6.21. *$cod(\varepsilon_1 \,⨟\, \varepsilon_2) = cod(\varepsilon_1) \,⨟\, cod(\varepsilon_2)$*

Proof. Direct by inspection on the inductive definition of consistent transitivity. □

Proposition 6.22. *$(\varepsilon_1 \,⨟\, \varepsilon_2)[E] = \varepsilon_1[E] \,⨟\, \varepsilon_2[E]$.*

Proof. Direct by inspection on the inductive definition of consistent transitivity. □

Lemma 6.23. *(Optimality of consistent transitivity).*
If $\varepsilon_3 = \varepsilon_1 \,⨟\, \varepsilon_2$ is defined, then $\pi_1(\varepsilon_3) \sqsubseteq \pi_1(\varepsilon_1)$ and $\pi_2(\varepsilon_3) \sqsubseteq \pi_2(\varepsilon_2)$.

Proof. Direct by inspection on the inductive definition of consistent transitivity. □

Lemma 6.24. *If $\varepsilon \vdash \Xi; \Delta \vdash G_1 \sim G_2$, $W \in \mathcal{S}[\![\Xi]\!]$ and $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$ then $\varepsilon_i^\rho \vdash W.\Xi_i; \Delta \vdash \rho(G_1) \sim \rho(G_2)$, where $\varepsilon_i^\rho = \rho_i(\varepsilon)$.*

Proof. Direct by induction on the structure of the types G_1 and G_2. □

Lemma 6.25. *If $\Xi; \Delta; \Gamma \vdash t : G$, $W \in \mathcal{S}[\![\Xi]\!]$, $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$ and $(W, \gamma) \in \mathcal{G}_\rho[\![\Gamma]\!]$ then $W.\Xi_i \vdash \rho(\gamma_i((t)) : \rho(G)$.*

Proof. Direct by induction on the structure of the term. □

Lemma 6.26. *If*
- $\varepsilon_i \Vdash W.\Xi_i \vdash G_i \sim \rho(G)$, $\varepsilon_1 \equiv \varepsilon_2$
- $\varepsilon \Vdash \Xi; \Delta \vdash G \sim G'$
- $W \in \mathcal{S}[\![\Xi]\!]$, $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$
- $\forall \alpha \in \Xi. \alpha^{E_i^*} \in p_2(\varepsilon_i) \Rightarrow E_1^* \equiv E_2^*$

then $\varepsilon_1 \,⨟\, \rho_1(\varepsilon) \iff \varepsilon_2 \,⨟\, \rho_2(\varepsilon)$.

Proof. We proceed by induction on the judgment $\varepsilon_i \vdash W.\Xi_i \vdash G_i \sim G$.

Case ($\varepsilon_i = \langle B_i, B_i \rangle$). Then the result is trivial as by definition of $\varepsilon_1 \equiv \varepsilon_2$, $B_1 = B_2$, therefore $\varepsilon_1 = \varepsilon_2$. As ε cannot have free type variables (otherwise the result holds immediately), proving that $\varepsilon_1 \,⨟\, \varepsilon \iff \varepsilon_1 \,⨟\, \varepsilon$ is trivial.

Case ($\varepsilon_i = \langle ?, ? \rangle$). As the combination with $\langle ?, ? \rangle$ never produce runtime errors, the result follows immediately as both operation never fail.

Case ($\varepsilon_i = \langle E_{1i}, \alpha^{E_{2i}} \rangle$). We branch on two sub cases:

- Case $\alpha \in \Xi$. Then ε has to have the form $\langle \alpha^{E_3}, E_4 \rangle$, $\langle ?, ? \rangle$ or $\langle ?, \beta^{\cdots^?} \rangle$ (otherwise the transitivity operator will always fails in both branches). Also E_4 cannot be a type variable X for instance, because X is consistent with only X or ?, and in either case the evidence gives you X on both sides of the evidence. And α cannot point to a type variable by construction (e.g, type α^X does not exists). Then ε cannot have free type variables, therefore $\rho_i(\varepsilon) = \varepsilon$, and therefore we have to prove: $\varepsilon_1 \,⨟\, \varepsilon \iff \varepsilon_2 \,⨟\, \varepsilon$. For cases where $\varepsilon = \langle ?, ? \rangle$ or $\varepsilon = \langle ?, \beta^{\cdots^?} \rangle$, then as they never produce runtime errors, the result follows immediately as both operation never fail. The interesting case is $\varepsilon = \langle \alpha^{E_3}, E_4 \rangle$. By definition of transitivity $\langle E_{1i}, \alpha^{E_{2i}} \rangle \,⨟\, \langle \alpha^{E_3}, E_4 \rangle = \langle E_{1i}, E_{2i} \rangle \,⨟\, \langle E_3, E_4 \rangle$. By Lemma 6.29, $\langle E_{1i}, E_{2i} \rangle \vdash W.\Xi_i \vdash G_i \sim \Xi(\alpha)$ and $\langle E_3, E_4 \rangle \vdash W.\Xi_i \vdash \Xi(\alpha) \sim G'$. Also we know by premise that $E_{2i} \equiv E_{2i}$, then by induction hypothesis $\langle E_{11}, E_{21} \rangle \,⨟\, \langle E_3, E_4 \rangle \iff \langle E_{12}, E_{22} \rangle \,⨟\, \langle E_3, E_4 \rangle$, and the result follows immediately.

- Case $\alpha \notin \Xi$. In this case ε has to have the form $\langle X, X\rangle$ (where $\rho_i(\varepsilon) = \langle lift_{W.\Xi_i}(\alpha), lift_{W.\Xi_i}(\alpha)\rangle$), $\langle ?, ?\rangle$ or $\langle ?, \beta^{\cdots^?}\rangle$, (otherwise the transitivity always fail in both cases). For cases where $\varepsilon = \langle ?, ?\rangle$ or $\varepsilon = \langle ?,\ \beta^{\cdots^?}\rangle$, by the definition of transitivity, they never produce runtime errors, then the result follows immediately as both operation never fail.
If $\varepsilon = \langle X, X\rangle$, by construction of evidence, $\alpha^{E_{2i}} \sqsubseteq lift_{W.\Xi_i}(\alpha) \sqsubseteq ?$, then by Lemma 6.30, we know that $\varepsilon_i \mathbin{;} \rho_i(\varepsilon) = \varepsilon_i$, and the result holds.

Case ($\varepsilon_i = \langle \alpha^{E_{i1}}, E_{i2}\rangle$). Then ε has the form $\langle E_3, E_4\rangle$, where $\rho_i(\varepsilon) = \langle E_{i3}, E_{i4}\rangle$. By the definition of transitivity we know that:

$$\langle \alpha^{E_{i1}, E_{i2}}\rangle \mathbin{;} \langle E_{i3}, E_{i4}\rangle \iff \langle E_{i1}, E_{i2}\rangle \mathbin{;} \langle E_{i3}, E_{i4}\rangle$$

Then by the induction hypothesis with:

$$\langle E_{i1}, E_{i2}\rangle \Vdash W.\Xi_i \vdash W.\Xi_i(\alpha) \sim \rho(G)$$

$$\varepsilon \Vdash \Xi; \Delta \vdash G \sim G'$$

we know that:

$$\langle E_{11}, E_{22}\rangle \mathbin{;} \langle E_{13}, E_{14}\rangle \iff \langle E_{21}, E_{22}\rangle \mathbin{;} \langle E_{23}, E_{24}\rangle$$

Then the result follows immediately.

Case ($\varepsilon_i = \langle E_{11i} \rightarrow E_{12i}, E_{21i} \rightarrow E_{22i}\rangle$). We analyze cases for ε:

- Case $\varepsilon = \langle ?, ?\rangle$ or $\varepsilon = \langle ?, \beta^{\cdots^?}\rangle$, then transitivity never fails as explained in previous cases.
- Case $\varepsilon = \langle E_{31} \rightarrow E_{32}, E_{41} \rightarrow E_{42}\rangle$. Then $\rho_i(\varepsilon) = \langle E_{31i} \rightarrow E_{32i}, E_{41i} \rightarrow E_{42i}\rangle$. By definition of interior and meet, the definition of transitivity for functions, can be rewritten like this:

$$\frac{\langle E_{41i}, E_{31i}\rangle \mathbin{;} \langle E_{21i}, E_{11i}\rangle = \langle E_{i3}, E_{i1}\rangle \qquad \langle E_{12i}, E_{22i}\rangle \mathbin{;} \langle E_{32i}, E_{42i}\rangle = \langle E_{i2}, E_{i4}\rangle}{\langle E_{11i} \rightarrow E_{12i}, E_{21i} \rightarrow E_{22i}\rangle \mathbin{;} \langle E_{31i} \rightarrow E_{32i}, E_{41i} \rightarrow E_{42i}\rangle = \langle E_{i1} \rightarrow E_{i2}, E_{i3} \rightarrow E_{i4}\rangle}$$

Also notice as the definition of interior is symmetrical (as consistency is symmetric), $\langle E_{41i}, E_{31i}\rangle \mathbin{;} \langle E_{21i}, E_{11i}\rangle = \langle E_{i3}, E_{i1}\rangle$ can be computed as $\langle E_{11i}, E_{21i}\rangle \mathbin{;} \langle E_{31i}, E_{41i}\rangle = \langle E_{i1}, E_{i3}\rangle$. Also $\varepsilon_1 \equiv \varepsilon_2$ implies that $dom(\varepsilon_1) \equiv dom(\varepsilon_2)$ and $cod(\varepsilon_1) \equiv cod(\varepsilon_2)$. And that $dom(\varepsilon) \Vdash \Xi; \Delta \vdash dom(G') \sim dom(G)$ is equivalent to:

$$\langle \pi_2(dom(\varepsilon)), \pi_1(dom(\varepsilon))\rangle \Vdash \Xi; \Delta \vdash dom(G) \sim dom(G')$$

where $cod(\varepsilon) \Vdash \Xi; \Delta \vdash cod(G) \sim cod(G')$. The result holds by applying induction hypothesis on:

$$\langle E_{11i}, E_{21i}\rangle \Vdash \Xi; \Delta \vdash dom(G_i) \sim dom(\rho(G))$$

$$\langle \pi_2(dom(\varepsilon)), \pi_1(dom(\varepsilon))\rangle \Vdash \Xi; \Delta \vdash dom(G) \sim dom(G')$$

and

$$\langle E_{12i}, E_{22i}\rangle \Vdash \Xi; \Delta \vdash cod(G_i) \sim cod(\rho(G))$$

$$cod(\varepsilon) \Vdash \Xi; \Delta \vdash cod(G) \sim cod(G')$$

- Case $\varepsilon = \langle E_{31} \rightarrow E_{32}, \alpha^{E_{41} \rightarrow E_{42}}\rangle$. Then $\rho_i(\varepsilon) = \langle E_{31i} \rightarrow E_{32i}, \alpha^{E_{41i} \rightarrow E_{42i}}\rangle$. We use a similar argument to the previous item noticing that

$$\frac{\dfrac{\langle E_{41i}, E_{31i}\rangle \mathbin{;} \langle E_{21i}, E_{11i}\rangle = \langle E_{i3}, E_{i1}\rangle \qquad \langle E_{12i}, E_{22i}\rangle \mathbin{;} \langle E_{32i}, E_{42i}\rangle = \langle E_{i2}, E_{i4}\rangle}{\langle E_{11i} \rightarrow E_{12i}, E_{21i} \rightarrow E_{22i}\rangle \mathbin{;} \langle E_{31i} \rightarrow E_{32i}, E_{41i} \rightarrow E_{42i}\rangle = \langle E_{i1} \rightarrow E_{i2}, E_{i3} \rightarrow E_{i4}\rangle}}{\langle E_{11i} \rightarrow E_{12i}, E_{21i} \rightarrow E_{22i}\rangle \mathbin{;} \langle E_{31} \rightarrow E_{32}, \alpha^{E_{41} \rightarrow E_{42}}\rangle = \langle E_{i1} \rightarrow E_{i2}, \alpha^{E_{i3} \rightarrow E_{i4}}\rangle}$$

and that if $G' = \alpha$ by Lemma 6.29

$$\frac{\langle E_{31} \rightarrow E_{32}, E_{41} \rightarrow E_{42}\rangle \vdash \Xi; \Delta \vdash G \sim \Xi(\alpha)}{\langle E_{31} \rightarrow E_{32}, \alpha^{E_{41}\rightarrow E_{42}}\rangle \vdash \Xi; \Delta \vdash G \sim \alpha}$$

and if $G' = ?$ by Lemma 6.29

$$\frac{\langle E_{31} \rightarrow E_{32}, E_{41} \rightarrow E_{42}\rangle \vdash \Xi; \Delta \vdash G \sim ?}{\langle E_{31} \rightarrow E_{32}, \alpha^{E_{41}\rightarrow E_{42}}\rangle \vdash \Xi; \Delta \vdash G \sim ?}$$

Case ($\varepsilon_i = \langle \forall X.E_{1i}, \forall X.E_{2i}\rangle$).

We proceed similar to the function case using induction hypothesis on the subtypes.

Case ($\varepsilon_i = \langle E_{1i} \times E_{2i}, E_{3i} \times E_{4i}\rangle$).

We proceed similar to the function case using induction hypothesis on the subtypes. □

Lemma 6.27. *If $\langle E_1, E_2\rangle \vdash \Xi; \Delta \vdash G_1 \sim G_2$, then*
(1) $\forall G_3$, unlift$(E_2) \sqsubseteq G_3 \sqsubseteq G_2$, $\langle E_1, E_2\rangle \vdash \Xi; \Delta \vdash G_1 \sim G_3$, and
(2) $\forall G_3$, unlift$(E_1) \sqsubseteq G_3 \sqsubseteq G_1$,$\langle E_1, E_2\rangle \vdash \Xi; \Delta \vdash G_3 \sim G_2$

Proof. By definition of evidence and interior noticing that always $E_i \sqsubseteq G_i$. □

Lemma 6.28. *If $\langle \alpha^{E_1}, E_2\rangle \vdash \Xi; \Delta \vdash \alpha \sim G$, then $\langle E_1, E_2\rangle \vdash \Xi; \Delta \vdash \Xi(\alpha) \sim G$.*

Proof. Direct by definition of interior and evidence. □

Lemma 6.29. *If $\langle E_1, \alpha^{E_2}\rangle \vdash \Xi; \Delta \vdash G \sim \alpha$, then $\langle E_1, E_2\rangle \vdash \Xi; \Delta \vdash G \sim \Xi(\alpha)$.*

Proof. Direct by definition of interior and evidence. □

Lemma 6.30. *If $E_2 \sqsubseteq E_3$ then $\langle E_1, E_2\rangle \mathbin{⨟} \langle E_3, E_3\rangle = \langle E_1, E_2\rangle$.*

Proof. We proceed by induction on $\langle E_1, E_2\rangle$. If $\langle E_3, E_3\rangle = \langle ?, ?\rangle$ by definition of transitivity the result holds immediately so we do not consider this case in the following.

Case ($\langle E_1, E_2\rangle = \langle ?, ?\rangle$). Then we know that $E_3 = ?$, and the result follows immediately.

Case ($\langle E_1, E_2\rangle = \langle E_1, \alpha^{E'_2}\rangle$). Then $\langle E_3, E_3\rangle = \langle \alpha^{E'_3}, \alpha^{E'_3}\rangle$. Then $\langle E_1, \alpha^{E'_2}\rangle \mathbin{⨟} \langle \alpha^{E'_3}, \alpha^{E'_3}\rangle$ boils down to $\langle E_1, E'_2\rangle \mathbin{⨟} \langle E'_3, E'_3\rangle$, if $E'_2 = \beta^{E''_2}$, then E'_3 has to be $\beta^{E''_3}$ and we repeat this process. Let us assume that $E'_2 \notin$ SITypeName, then by definition of meet $E'_3 \notin$ SITypeName. By definition of precision if $\alpha^{E'_2} \sqsubseteq \alpha^{E'_3}$, then $E'_2 \sqsubseteq E'_3$. Then by induction hypothesis $\langle E_1, E'_2\rangle \mathbin{⨟} \langle E'_3, E'_3\rangle = \langle E_1, E'_2\rangle$, then $\langle E_1, \alpha^{E'_2}\rangle \mathbin{⨟} \langle \alpha^{E'_3}, \alpha^{E'_3}\rangle = \langle E_1, \alpha^{E'_2}\rangle$ and the result holds.

Case ($\langle E_1, E_2\rangle = \langle \alpha^{E'_1}, E_2\rangle$). Then $\langle \alpha^{E'_1}, E_2\rangle \mathbin{⨟} \langle E_3, E_3\rangle$ boils down to $\langle E'_1, E_2\rangle \mathbin{⨟} \langle E_3, E_3\rangle$. We know that $E_2 \sqsubseteq E_3$. Then by induction hypothesis $\langle E'_1, E_2\rangle \mathbin{⨟} \langle E_3, E_3\rangle = \langle E_1, E'_2\rangle$, then $\langle \alpha^{E'_1}, E_2\rangle \mathbin{⨟} \langle E_3, E_3\rangle = \langle \alpha^{E'_1}, E_2\rangle$ and the result holds.

Case ($\langle E_1, E_2\rangle = \langle B, B\rangle$). Then by definition of precision E_3 is either ? (case we wont analyze) or B. But $\langle B, B\rangle \mathbin{⨟} \langle B, B\rangle = \langle B, B\rangle$ and the result holds.

Case ($\langle E_1, E_2\rangle = \langle E_{11} \rightarrow E_{12}, E_{21} \rightarrow E_{22}\rangle$). Then E_3 has to have the form $E_{31} \rightarrow E_{32}$. By definition of precision, if $E_{21} \rightarrow E_{22} \sqsubseteq E_{31} \rightarrow E_{32}$ then $E_{21} \sqsubseteq E_{31}$ and $E_{22} \sqsubseteq E_{32}$. As $\langle E_{31}, E_{31}\rangle \mathbin{⨟} \langle E_{21}, E_{11}\rangle = (\langle E_{11}, E_{21}\rangle \mathbin{⨟} \langle E_{31}, E_{31}\rangle)^{-1}$. By induction hypothesis $\langle E_{11}, E_{21}\rangle \mathbin{⨟} \langle E_{31}, E_{31}\rangle = \langle E_{11}, E_{21}\rangle$ and $\langle E_{12}, E_{22}\rangle \mathbin{⨟} \langle E_{32}, E_{32}\rangle = \langle E_{12}, E_{22}\rangle$. Therefore $\langle E_{11} \rightarrow E_{12}, E_{21} \rightarrow E_{22}\rangle \mathbin{⨟} \langle E_{31} \rightarrow E_{32}, E_{31} \rightarrow E_{32}\rangle = \langle E_{11} \rightarrow E_{12}, E_{21} \rightarrow E_{22}\rangle$ and the result holds.

$$
\begin{array}{rcll}
C & ::= & [\cdot] \mid \varepsilon C_u :: G \mid \langle C, t\rangle \mid \langle t, C\rangle \mid C\ t \mid t\ C \mid \varepsilon C :: G \mid op(\overline{t}, C, \overline{t}) \mid C\ [G] \mid \pi_i(C) & \text{(GSF}\varepsilon\text{ Contexts)}\\
C_u & ::= & \lambda x : G.C \mid \Lambda X.C \mid \langle C_u, u\rangle \mid \langle u, C_u\rangle & \\
C_s & ::= & C \mid C_u &
\end{array}
$$

$\boxed{\vdash C : (\Xi; \Delta; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash G')}$ **Well-typed contexts**

$$\text{(Cid)}\frac{\Xi \subseteq \Xi' \quad \Delta \subseteq \Delta' \quad \Gamma \subseteq \Gamma' \quad \Xi; \Delta \vdash \Gamma \quad \Xi'; \Delta' \vdash \Gamma'}{\vdash [\cdot] : (\Xi; \Delta; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash G)}$$

$$\text{(C}\lambda\text{)}\frac{\vdash C : (\Xi; \Delta; \Gamma, x : G_1 \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma', x : G_1 \vdash G_2)}{\vdash \lambda x : G_1.C : (\Xi; \Delta; \Gamma, x : G_1 \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash G_1 \to G_2)}$$

$$\text{(C}\Lambda\text{)}\frac{\vdash C : (\Xi; \Delta, X; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta', X; \Gamma' \vdash G') \quad \Xi; \Delta \vdash \Gamma \quad \Xi'; \Delta' \vdash \Gamma'}{\vdash \Lambda X.C : (\Xi; \Delta, X; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash \forall X.G')}$$

$$\text{(CpairL)}\frac{\vdash C : (\Xi; \Delta; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash G_1) \quad \Xi'; \Delta'; \Gamma' \vdash t : G_2}{\vdash \langle C, t\rangle : (\Xi; \Delta; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash G_1 \times G_2)}$$

$$\text{(CpairR)}\frac{\Xi'; \Delta'; \Gamma' \vdash t : G_1 \quad \vdash C : (\Xi; \Delta; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash G_2)}{\vdash \langle t, C\rangle : (\Xi; \Delta; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash G_1 \times G_2)}$$

$$\text{(Casc)}\frac{\vdash C_s : (\Xi; \Delta; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash G') \quad \varepsilon \Vdash \Xi; \Delta \vdash G' \sim G''}{\vdash \varepsilon C_s :: G'' : (\Xi; \Delta; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash G'')}$$

$$\text{(Cop)}\frac{\begin{array}{ll}\Xi'; \Delta'; \Gamma' \vdash \overline{t_1} : \overline{G_1} & \vdash C : (\Xi; \Delta; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash G_3)\\ \Xi'; \Delta'; \Gamma' \vdash \overline{t_2} : \overline{G_2} & ty(op) = (\overline{G_1}, G_3, \overline{G_2}) \to G''\end{array}}{\vdash op(\overline{t_1}, C, \overline{t_2}) : (\Xi; \Delta; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash G'')}$$

$$\text{(CappL)}\frac{\vdash C : (\Xi; \Delta; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash G_1 \to G_2) \quad \Xi'; \Delta'; \Gamma' \vdash t : G_1}{\vdash C\ t : (\Xi; \Delta; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash G_2)}$$

$$\text{(CappR)}\frac{\Xi'; \Delta'; \Gamma' \vdash t : G_1 \to G_2 \quad \vdash C : (\Xi; \Delta; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash G_1)}{\vdash t\ C : (\Xi; \Delta; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash G_2)}$$

$$\text{(CappG)}\frac{\vdash C : (\Xi; \Delta; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash \forall X.G') \quad \Xi'; \Delta' \vdash G''}{\vdash C\ [G''] : (\Xi; \Delta; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash G'[G''/X])}$$

$$\text{(Cpair}i\text{)}\frac{\vdash C : (\Xi; \Delta; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash G_1 \times G_2)}{\vdash \pi_i(C) : (\Xi; \Delta; \Gamma \vdash G) \rightsquigarrow (\Xi'; \Delta'; \Gamma' \vdash G_i)}$$

Fig. 24. GSFε: Syntax and Static Semantics - Contexts

Case ($\langle E_1, E_2\rangle = \langle \forall X.E_{11}, \forall X.E_{21}\rangle$ or $\langle E_1, E_2\rangle = \langle E_{11} \times E_{12}, E_{21} \times E_{22}\rangle$). Analogous to function case.

□

6.3 Contextual Equivalence

In this section we show that the logical relation is sound with respect to contextual approximation (and therefore contextual equivalence). Figure 24 presents the syntax and static semantics of contexts.

Definition 6.31 (Contextual Approximation and Equivalence).

$$\begin{aligned}\Xi;\Delta;\Gamma \vdash t_1 \preceq^{ctx} t_2 : G \triangleq\ & \Xi;\Delta;\Gamma \vdash t_1 : G \wedge \Xi;\Delta;\Gamma \vdash t_2 : G \wedge \forall C, \Xi', G'. \\ & \vdash C : (\Xi;\Delta;\Gamma \vdash G) \rightsquigarrow (\Xi';\cdot;\cdot \vdash G') \Rightarrow \big((\Xi' \triangleright t_1 \Downarrow \implies \Xi' \triangleright t_2 \Downarrow) \wedge \\ & (\exists \Xi_1.\Xi' \triangleright C[t_1] \longmapsto^* \Xi_1 \triangleright \mathbf{error} \Rightarrow \exists \Xi_2.\Xi' \triangleright C[t_2] \longmapsto^* \Xi_2 \triangleright \mathbf{error})\big)\end{aligned}$$

$$\Xi;\Delta;\Gamma \vdash t_1 \approx^{ctx} t_2 : G \triangleq \Xi;\Delta;\Gamma \vdash t_1 \preceq^{ctx} t_2 : G \wedge \Xi;\Delta;\Gamma \vdash t_2 \preceq^{ctx} t_1 : G$$

Theorem 6.32 (Soundness w.r.t. Contextual Approximation). *If $\Xi;\Delta;\Gamma \vdash t_1 \preceq t_2 : G$ then $\Xi;\Delta;\Gamma \vdash t_1 \preceq^{ctx} t_2 : G$.*

Proof. The proof follows the usual route of going through congruence and adequacy. ◻

7 PARAMETRICITY VS. THE DGG IN GSF

In this section, we present the proofs of the auxiliary Lemmas need to show that the definition of parametricity for GSF is incompatible with the DGG.

Lemma 10.6. *Let* $\vdash (\Lambda X.\lambda x : ?.t) \rightsquigarrow v_a : \forall X.? \to X$ *and* $\vdash v \rightsquigarrow v_b : ?$. *For any* G_1 *and* G_2*, such that* $const(G_1) \neq const(G_2)$*, if* $\cdot \triangleright v_a\ [G_i] \longmapsto \alpha := G_i \triangleright \varepsilon_i v_i :: ? \to G_i$, $\varepsilon_i \Vdash ? \to \alpha \sim ? \to G_i$ *then* $\forall W \in \mathcal{S}[\![\cdot]\!], \forall R \in \textsc{Rel}_{W.j}[G_1, G_2], (W \boxtimes (\alpha, G_1, G_2, R), dom(\varepsilon_1)v_b :: ?, dom(\varepsilon_2)v_b :: ?) \in \mathcal{T}_{X\mapsto\alpha}[\![?]\!]$

Proof. Notice that v_a has to be of the form $(\varepsilon'(\Lambda X.\varepsilon''(\lambda x : ?.t') :: ? \to X) :: \forall X.? \to X)$, where $\varepsilon' = \langle \forall X.? \to X, \forall X.? \to X\rangle$ and $\varepsilon'' = \langle ? \to X, ? \to X\rangle$. Then $\cdot \triangleright v_a\ [G_i] \longmapsto \langle ? \to \hat{\alpha}_i, ? \to E_i\rangle t'$ for some t', where $\hat{\alpha}_i = lift_{\alpha\mapsto G_i}(\alpha)$ and $E_i = lift_{\cdot}(G_i)$. We know that $\cdot;\cdot;\cdot \vdash v_b : ?$ then as $X \notin FTV(v)$, $\cdot;X;\cdot \vdash v_b : ?$, therefore by the fundamental property (Thm 10.1), $\cdot;X;\cdot \vdash v_b \preceq v_b : ?$, therefore as $W \in \mathcal{S}[\![\cdot]\!]$, we can pick $W' = W \boxtimes (\alpha, G_1, G_2, R) \in \mathcal{S}[\![\cdot]\!]$, and $(W', X \mapsto \alpha) \in \mathcal{D}[\![X]\!]$ and thus conclude that $(W', v_b, v_b) \in \mathcal{T}_{X\mapsto\alpha}[\![?]\!]$. Now notice that $dom(\varepsilon_i) = \langle ?, ?\rangle$, but $\varepsilon \mathbin{⨟} \langle ?, ?\rangle = \varepsilon$ for any evidence ε, therefore $\alpha := G_i \triangleright dom(\varepsilon_i)v_b :: ? \longmapsto \alpha := G_i \triangleright v_b$, then we have to prove that $(\downarrow W', v_b, v_b) \in \mathcal{T}_{X\mapsto\alpha}[\![?]\!]$ which follows directly from the weakening lemma. ◻

Lemma 10.7. *For any* $\vdash v : ?$ *and* $\vdash G$*, we have* $(\Lambda X.\lambda x : ?.x :: X)\ [G]\ v \Downarrow$ **error**.

Proof. Let $id_? \triangleq \Lambda X.\lambda x : ?.x :: X$, $\vdash id_? \rightsquigarrow v_a : \forall X.? \to X$, and v s.t. $\vdash v \rightsquigarrow v_b : ?$.

By the fundamental property (Th. 10.1), $\vdash v_a \preceq v_a : \forall X.? \to X$ so for any $W_0 \in \mathcal{S}[\![\cdot]\!]$, $(W_0, v_a, v_a) \in \mathcal{T}_\emptyset[\![\forall X.? \to X]\!]$. Because v_a is a value, $(W_0, v_a, v_a) \in \mathcal{V}_\emptyset[\![\forall X.? \to X]\!]$. By reduction, $\cdot \triangleright v_a\ [G_i] \longmapsto^* \Xi'_i \triangleright \varepsilon'_i v_i :: ? \to G_i$ for some ε'_i, ε_i and $\varepsilon_{i\alpha}$, where $\Xi'_i = \{\alpha := G_i\}$ and $v_i = \varepsilon_i(\lambda x : ?.(\varepsilon_{i\alpha} x :: \alpha)) :: ? \to \alpha$. We can instantiate the definition of $\mathcal{V}_\emptyset[\![\forall X.? \to X]\!]$ with W_0, $G_1 = G$ and G_2 structurally different (and different from ?), some $R \in \textsc{Rel}_{W_0.j}[G_1, G_2]$, v_1, v_2, ε'_1 and ε'_2, then we have that $(W_1, v_1, v_2) \in \mathcal{T}_{X\mapsto\alpha}[\![? \to X]\!]$, where $W_1 = (\downarrow(W_0 \boxtimes (\alpha, G_1, G_2, R))$. As v_1 and v_2 are values, $(W_1, v_1, v_2) \in \mathcal{V}_{X\mapsto\alpha}[\![? \to X]\!]$. Also, by associativity of consistent transitivity, the reduction of $\Xi'_i \triangleright (\varepsilon'_i v_i :: ? \to G_i)\ v_?$ is equivalent to that of $\Xi'_i \triangleright cod(\varepsilon'_i)(v_i\ (dom(\varepsilon'_i)v_? :: ?)) :: G_i$.

By the fundamental property (Th. 10.1) we know that $\vdash v_b \preceq v_b : ?$; we can instantiate this definition with W_0, and we have that $(W_0, v_b, v_b) \in \mathcal{V}_\emptyset[\![?]\!]$. By Lemma 10.6, $(W_1, dom(\varepsilon'_1)v_? :: ?, dom(\varepsilon'_2)v_? :: ?) \in \mathcal{T}_{X\mapsto\alpha}[\![?]\!]$. If $dom(\varepsilon'_1)v_? :: ?$ reduces to **error** then the result follows immediately. Otherwise, $\Xi'_i \triangleright dom(\varepsilon'_1)v_? :: ? \longmapsto^* \Xi'_i \triangleright v''_i$, and $(W_2, v''_1, v''_2) \in \mathcal{V}_{X\mapsto\alpha}[\![?]\!]$, where $W_2 =\downarrow W_1$, and some v''_1 and v''_2. We can instantiate the definition of $\mathcal{V}_{X\mapsto\alpha}[\![? \to X]\!]$ with W_2, v''_1 and v''_2, obtaining that $(W_2, v_1\ v''_1, v_2\ v''_2) \in \mathcal{T}_{X\mapsto\alpha}[\![X]\!]$. We then proceed by contradiction. Suppose that $\Xi'_i \triangleright v_i\ v''_i \longmapsto^* \Xi''_i \triangleright v'_i$ (for a big-enough step index). If $v''_i = \varepsilon''_{iv} u :: ?$, then by evaluation $v'_i = \varepsilon'_{iv} u :: \alpha$, for some ε'_{iv}. But by definition of $\mathcal{V}_{X\mapsto\alpha}[\![X]\!]$, it must be the case that for some $W_3 \succeq W_2$, $(W_3, \varepsilon'_{1v} u :: G_1, \varepsilon'_{2v} u :: G_2) \in R$, which is impossible because u cannot be ascribed to structurally different types G_1 and G_2. Therefore $v_1\ v''_1$ cannot reduce to a value, and hence the term $v_a\ [G]\ v_b$ cannot reduce to a value either. Because v_a is non-diverging, its application must produce **error**. ◻

8 A CHEAP THEOREM IN GSF

This section shows the proof of the cheap theorem presented in the paper and some auxiliary results.

Definition 8.1. Let $\mathcal{X}(t, \alpha)$ a predicate that holds if and only if in each evidence of term t, if α is present, then it appears on both sides of the evidence and in the same structural position. This predicate is defined inductively as follows:

$$\frac{\forall \varepsilon \in t, \mathcal{X}(\varepsilon, \alpha)}{\mathcal{X}(t, \alpha)}$$

where

$$\frac{}{\mathcal{X}(\langle \alpha^E, \alpha^E \rangle, \alpha)} \qquad \frac{\alpha \notin FTN(E_1) \cup FTN(E_2)}{\mathcal{X}(\langle E_1, E_2 \rangle, \alpha)} \qquad \frac{\mathcal{X}(\langle E_1, E_3 \rangle, \alpha) \qquad \mathcal{X}(\langle E_2, E_4 \rangle, \alpha)}{\mathcal{X}(\langle E_1 \to E_2, E_3 \to E_4 \rangle, \alpha)}$$

$$\frac{\mathcal{X}(\langle E_1, E_3 \rangle, \alpha) \qquad \mathcal{X}(\langle E_2, E_4 \rangle, \alpha)}{\mathcal{X}(\langle E_1 \times E_2, E_3 \times E_4 \rangle, \alpha)} \qquad \frac{\mathcal{X}(\langle E_1, E_2 \rangle, \alpha)}{\mathcal{X}(\langle \forall X.E_1, \forall X.E_2 \rangle, \alpha)}$$

Corollary 10.9. *Let t and v be static terms such that $\vdash t : \forall X.T$, $\vdash v : T'$, and $t[T']\ v \Downarrow v'$.*
(1) If $\forall X.T \sqsubseteq \forall X.X \to ?$ then $(t :: \forall X.X \to ?)[T']\ v \Downarrow v''$, and $v' \leqslant v''$.
(2) If $\forall X.T \sqsubseteq \forall X.? \to X$ then $(t :: \forall X.? \to X)[T']\ v \Downarrow v''$, and $v' \leqslant v''$.

Proof. Direct by Lemmas 9.4 and 9.7. □

Lemma 8.2. *$\forall W \in \mathcal{S}[\![\Xi]\!], \rho, \gamma.((W, \rho) \in \mathcal{D}[\![\Delta]\!] \wedge (W, \gamma) \in \mathcal{G}_\rho[\![\Gamma]\!])$, such that $\forall v \in cod(\gamma_i), \mathcal{X}(v, \alpha)$. If $\mathcal{X}(\rho(\gamma_i(t_i)), \alpha)$, then $\Xi \triangleright \rho(\gamma_i(t_i)) \longmapsto \Xi' \triangleright t_i'$ and $\mathcal{X}(t', \alpha)$*

Proof. By induction on the structure of t_i. The proof is direct by looking at the inductive definition of construction of evidences (interior), noticing that $\forall G, \mathcal{I}(X, G) = \mathcal{I}(G, X) = \langle X, X \rangle$. Then by inspection of consistent transitivity we know that, for any evidence of a value $\langle E_1, E_2 \rangle$

$$\langle E_1, E_2 \rangle \,;\, \langle \alpha^E, \alpha^E \rangle = \langle E_1', \alpha^{E'} \rangle \wedge E_1' \neq \alpha^* \iff E_2 = \alpha^{E''} \wedge E_1 \neq \alpha^*$$

but if that is the case $\neg(\mathcal{X}(\langle E_1, E_2 \rangle, \alpha))$, which contradicts the premise. □

Theorem 10.10. *Let $v \triangleq \Lambda X.\lambda x : ?.t$ for some t, such that $\vdash v : \forall X.? \to X$. Then for any $\vdash v' : G$, we either have $v\ [G]\ v' \Downarrow$ **error** or $v\ [G]\ v' \Uparrow$.*

Proof. Let $\vdash v \rightsquigarrow v_\forall : \forall X.? \to X$, $\vdash v' \rightsquigarrow v_? : ?$. Because $\vdash v_\forall : \forall X.? \to X$ and $\vdash v_? : ?$, by the fundamental property (Theorem 10.1) we know that

$$(W_0, v_\forall, v_\forall) \in \mathcal{V}_\emptyset[\![\forall X.? \to X]\!]$$

$$(W_0, v_?, v_?) \in \mathcal{V}_\emptyset[\![?]\!]$$

Let $v_\forall = \varepsilon(\Lambda X.(\lambda x : ?.t)) :: \forall X.? \to X$, where $\varepsilon \Vdash \cdot; \cdot \vdash \forall X.? \to X \sim \forall X.? \to X$, and therefore $\varepsilon = \langle \forall X.? \to X, \forall X.? \to X \rangle$.

Note that by the reduction rules we know that

$$\Xi \triangleright v_\forall\ [G] \longmapsto^* \Xi_1' \triangleright \varepsilon_1(\varepsilon_2(\lambda x : ?.t') :: ? \to \alpha) :: ? \to G$$

for some t', where $\varepsilon_1 = \langle ? \to \alpha^E, ? \to E \rangle$, $\varepsilon_2 = \langle ? \to \alpha^E, ? \to \alpha^E \rangle$, $E = lift_\cdot(G)$, $\Xi_1' = \Xi$, $\alpha = G$.

By definition of $\mathcal{V}_\emptyset[\![\forall X.? \to X]\!]$ if we pick $G_1 = G_2 = G$, and some R, then for some W_1 we know that $(W_1, v_1, v_2) \in \mathcal{V}_{X \mapsto \alpha}[\![? \to X]\!]$, where $v_i = \varepsilon_2(\lambda x : ?.t') :: ? \to \alpha$.

Also, by the reduction rules we know that $\Xi_i' \triangleright (\varepsilon_1 v_i :: ? \to G)\ v_? \iff \Xi_i' \triangleright cod(\varepsilon_1)(v_i\ (dom(\varepsilon_1) v_? :: ?)) :: G$. As $dom(\varepsilon_1) = \langle ?, ? \rangle$, then $\Xi' \triangleright dom(\varepsilon_1) v_? :: ? \longmapsto \Xi' \triangleright v_? :: ?$. As $\alpha \notin FTN(v_?)$, then $\mathcal{X}(v_?, \alpha)$.

Also we know that $\mathcal{X}(v_i, \alpha)$. Then by Lemma 8.2, if $\Xi' \triangleright t'[v_?] \longmapsto^* v'$, then $\mathcal{X}(v', \alpha)$, but that is a contradiction because if $(W_4, v', v') \in \mathcal{V}_\rho[\![\alpha]\!]$, then $\neg\mathcal{X}(v', \alpha)$ and the result holds. $\square$

9 EMBEDDING DYNAMIC SEALING IN GSF

In this section, we prove Theorem 11.1, using the simulation relation $\approx$ between λ_{seal} and GSFε, defined in Figure 15. We also define a direct embedding of λ_{seal} into GSFε to make the proof simpler.

$$
\begin{aligned}
\lceil x \rceil_\varepsilon &= x \\
\lceil \sigma \rceil_\varepsilon &= su^{\sigma}_{\varepsilon} \\
\lceil b \rceil_\varepsilon &= \varepsilon_B(\varepsilon_B b :: B) :: ? \\
\lceil \lambda x.t \rceil_\varepsilon &= \varepsilon_{?\to?}(\varepsilon_{?\to?}\lambda x.\lceil t \rceil_\varepsilon :: ? \to ?) :: ? \\
\lceil \langle t_1, t_2 \rangle \rceil_\varepsilon &= \varepsilon_{?\times?}\langle \lceil t_1 \rceil_\varepsilon, \lceil t_2 \rceil_\varepsilon \rangle :: ? \\
\lceil \pi_i(t) \rceil_\varepsilon &= \pi_i(\varepsilon_{?\times?}\lceil t \rceil_\varepsilon :: ? \times ?) \\
\lceil op(\overline{t}) \rceil_\varepsilon &= \mathsf{let}\ \overline{x} : ? = \lceil \overline{t} \rceil\ \mathsf{in}\ \varepsilon_B\, op(\varepsilon_{\overline{B}}\overline{x} :: \overline{B}) :: ? \\
\lceil \nu x.t \rceil_\varepsilon &= \mathsf{let}\ x = su_\varepsilon\ \mathsf{in}\ \lceil t \rceil_\varepsilon \\
\lceil t_1\ t_2 \rceil_\varepsilon &= \mathsf{let}\ x = \lceil t_1 \rceil_\varepsilon\ \mathsf{in}\ \mathsf{let}\ y = \lceil t_2 \rceil_\varepsilon\ \mathsf{in}\ (\varepsilon_{?\to?}x :: ? \to ?)\ y \\
\lceil \{t_1\}_{t_2} \rceil_\varepsilon &= \mathsf{let}\ x = \lceil t_1 \rceil_\varepsilon\ \mathsf{in}\ \mathsf{let}\ y = \lceil t_2 \rceil_\varepsilon\ \mathsf{in}\ (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ? \times ?) :: ? \to ?)\ x \\
\lceil \mathsf{let}\ \{z\}_{t_1} = t_2\ \mathsf{in}\ t_3 \rceil_\varepsilon &= \mathsf{let}\ x = \lceil t_1 \rceil_\varepsilon\ \mathsf{in}\ \mathsf{let}\ y = \lceil t_2 \rceil_\varepsilon\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}x :: ? \times ?) :: ? \to ?\ y\ \mathsf{in}\ \lceil t_3 \rceil_\varepsilon
\end{aligned}
$$

Fig. 25. Compilation from λ_{seal} to GSFε

Definition 9.1. We said that μ and Ξ are synchronized, denoted $\mu \equiv \Xi$, if and only if $\sigma \in \mu \iff \sigma := ? \in \Xi$.

Lemma 9.2. *Let t be a λ_{seal} term. If $\Xi; \Gamma \vdash \lceil t \rceil \rightsquigarrow t_\varepsilon : ?$ then $\lceil t \rceil_\varepsilon = t_\varepsilon$.*

Proof. The proof is straightforward by induction on the syntax of t, and following definitions of $\lceil t \rceil$, $\Xi; \Gamma \vdash \lceil t \rceil \rightsquigarrow t_\varepsilon : ?$ and $\lceil t \rceil_\varepsilon$. □

Lemma 9.3. *If $\Xi; \Gamma \vdash \lceil t \rceil \rightsquigarrow t_\varepsilon : ?$, then $\mu; \Xi; \Gamma \vdash t \approx t_\varepsilon : ?$, for some $\mu \equiv \Xi$.*

Proof. By Lemma 9.2, we know that $t_\varepsilon = \lceil t \rceil_\varepsilon$. Therefore, we are required to prove that $\mu; \Xi; \Gamma \vdash t \approx \lceil t \rceil_\varepsilon : ?$. We follow by induction on the syntax of t. Since translation preserves typing (Theorem **??**), we know that $\Xi; \Gamma \vdash \lceil t \rceil_\varepsilon : ?$.

Case (x). Then, we know that

$$\lceil x \rceil_\varepsilon = x$$

We have $t = x$. By premise we know that $\Xi; \Gamma \vdash x : ?$ which implies that $x : ? \in \Gamma$ and $\Xi; \vdash \Gamma$. Therefore, $\mu; \Xi; \Gamma \vdash t \approx \lceil t \rceil_\varepsilon : ?$ by Rule (Rx) and the result follows immediately.

Case (b). Then, we know that

$$\lceil b \rceil_\varepsilon = \varepsilon_B(\varepsilon_B b :: B) :: ?$$

We have $t = b$. Then, we have to prove that $\mu; \Xi; \Gamma \vdash b \approx \varepsilon_B(\varepsilon_B b :: B) :: ? : ?$. We know by the Rule (Rb) that $\mu; \Xi; \Gamma \vdash b \approx \varepsilon_B b :: ? : ?$. Therefore, by the Rule (Ru) the result follows immediately.

Case ($\lambda x.t'$). Then, we know that

$$\lceil \lambda x.t' \rceil_\varepsilon = \varepsilon_{?\to?}(\varepsilon_{?\to?}\lambda x.\lceil t' \rceil_\varepsilon :: ? \to ?) :: ?$$

We have $t = \lambda x.t'$. Then, we have to prove that $\mu; \Xi; \Gamma \vdash \lambda x.t' \approx \varepsilon_{?\to?}(\varepsilon_{?\to?}\lambda x.\lceil t' \rceil_\varepsilon :: ? \to ?) :: ? : ?$. Since $\Xi; \Gamma \vdash \lceil t \rceil_\varepsilon : ?$ and by Lemma 9.13, we know that $\Xi; \Gamma, x : ? \vdash \lceil t' \rceil_\varepsilon : ?$, thus by the induction hypothesis $\mu; \Xi; \Gamma, x : ? \vdash t' \approx \lceil t' \rceil_\varepsilon : ?$. Therefore, by the Rule (Rλ) that $\mu; \Xi; \Gamma \vdash \lambda x.t' \approx \varepsilon_{?\to?}\lambda x.\lceil t' \rceil_\varepsilon :: ? : ?$. Therefore, by the Rule (Ru) the result follows immediately.

Case (σ). Then, we know that

$$\lceil \sigma \rceil_\varepsilon = su^\sigma_\varepsilon$$

We have $t = \sigma$. Then, we have to prove that $\mu; \Xi; \Gamma \vdash \sigma \approx su^\sigma_\varepsilon : ?$. By premise we know that $\Xi; \Gamma \vdash su^\sigma_\varepsilon : ?$ which implies that $\sigma := ? \in \Xi$ and $\Xi \vdash \Gamma$. Therefore, by the Rule (Rs) the result follows immediately.

Case ($t_1\ t_2$). Then, we know that

$$\lceil t_1\ t_2 \rceil_\varepsilon = \mathsf{let}\ x = \lceil t_1 \rceil_\varepsilon\ \mathsf{in}\ \mathsf{let}\ y = \lceil t_2 \rceil_\varepsilon\ \mathsf{in}\ (\varepsilon_{?\to ?} x :: ? \to ?)\ y$$

We have $t = t_1\ t_2$. Then, we have to prove that

$$\mu; \Xi; \Gamma \vdash t_1\ t_2 \approx \mathsf{let}\ x = \lceil t_1 \rceil_\varepsilon\ \mathsf{in}\ \mathsf{let}\ y = \lceil t_2 \rceil_\varepsilon\ \mathsf{in}\ (\varepsilon_{?\to ?} x :: ? \to ?)\ y : ?$$

Since $\Xi; \Gamma \vdash \lceil t \rceil_\varepsilon : ?$ and by Lemma 9.13, we know that $\Xi; \Gamma \vdash \lceil t_1 \rceil_\varepsilon : ?$ and $\Xi; \Gamma \vdash \lceil t_2 \rceil_\varepsilon : ?$. By the induction hypothesis, we know that $\mu; \Xi; \Gamma \vdash t_1 \approx \lceil t_1 \rceil_\varepsilon : ?$ and $\mu; \Xi; \Gamma \vdash t_2 \approx \lceil t_2 \rceil_\varepsilon : ?$. Therefore, by the Rule (RappL) the result follows immediately.

Case ($\pi_i(t')$). Then, we know that

$$\lceil \pi_i(t') \rceil_\varepsilon = \pi_i(\varepsilon_{?\times ?} \lceil t' \rceil_\varepsilon :: ? \times ?)$$

We have $t = \pi_i(t')$. Then, we have to prove that $\mu; \Xi; \Gamma \vdash \pi_i(t') \approx \pi_i(\varepsilon_{?\times ?} \lceil t' \rceil_\varepsilon :: ? \times ?) : ?$. Since $\Xi; \Gamma \vdash \lceil t \rceil_\varepsilon : ?$ and by Lemma 9.13, we know that $\Xi; \Gamma \vdash \lceil t' \rceil_\varepsilon : ?$. By the induction hypothesis, we know that $\mu; \Xi; \Gamma \vdash t' \approx \lceil t' \rceil_\varepsilon : ?$. Therefore, by the Rule (Rpi) the result follows immediately.

Case ($\{t_1\}_{t_2}$). Then, we know that

$$\lceil \{t_1\}_{t_2} \rceil_\varepsilon = \mathsf{let}\ x = \lceil t_1 \rceil_\varepsilon\ \mathsf{in}\ \mathsf{let}\ y = \lceil t_2 \rceil_\varepsilon\ \mathsf{in}\ (\varepsilon_{?\to ?} \pi_1(\varepsilon_{?\times ?} y :: ? \times ?) :: ? \to ?)\ x$$

We have $t = \{t_1\}_{t_2}$. Then, we have to prove that

$$\mu; \Xi; \Gamma \vdash \{t_1\}_{t_2} \approx \mathsf{let}\ x = \lceil t_1 \rceil_\varepsilon\ \mathsf{in}\ \mathsf{let}\ y = \lceil t_2 \rceil_\varepsilon\ \mathsf{in}\ (\varepsilon_{?\to ?} \pi_1(\varepsilon_{?\times ?} y :: ? \times ?) :: ? \to ?)\ x : ?$$

Since $\Xi; \Gamma \vdash \lceil t \rceil_\varepsilon : ?$ and by Lemma 9.13, we know that $\Xi; \Gamma \vdash \lceil t_1 \rceil_\varepsilon : ?$ and $\Xi; \Gamma \vdash \lceil t_2 \rceil_\varepsilon : ?$.By the induction hypothesis, we know that $\mu; \Xi; \Gamma \vdash t_1 \approx \lceil t_1 \rceil_\varepsilon : ?$ and $\mu; \Xi; \Gamma \vdash t_2 \approx \lceil t_2 \rceil_\varepsilon : ?$. Therefore, by the Rule (Rsed1L) the result follows immediately.

Case ($\mathsf{let}\ \{x\}_{t_1} = t_2\ \mathsf{in}\ t_3$). Then, we know that

$$\lceil \mathsf{let}\ \{x\}_{t_1} = t_2\ \mathsf{in}\ t_3 \rceil_\varepsilon = \mathsf{let}\ x = \lceil t_1 \rceil_\varepsilon\ \mathsf{in}\ \mathsf{let}\ y = \lceil t_2 \rceil_\varepsilon\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to ?} \pi_2(\varepsilon_{?\times ?} x :: ? \times ?) :: ? \to ?\ y\ \mathsf{in}\ \lceil t_3 \rceil_\varepsilon$$

We have $t = \mathsf{let}\ \{x\}_{t_1} = t_2\ \mathsf{in}\ t_3$. Then, we have to prove that

$$\mu; \Xi; \Gamma \vdash \mathsf{let}\ \{x\}_{t_1} = t_2\ \mathsf{in}\ t_3 \approx \mathsf{let}\ x = \lceil t_1 \rceil_\varepsilon\ \mathsf{in}\ \mathsf{let}\ y = \lceil t_2 \rceil_\varepsilon\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to ?} \pi_2(\varepsilon_{?\times ?} x :: ? \times ?) :: ? \to ?\ y\ \mathsf{in}\ \lceil t_3 \rceil_\varepsilon : ?$$

Since $\Xi; \Gamma \vdash \lceil t \rceil_\varepsilon : ?$ and by Lemma 9.13, we know that $\Xi; \Gamma \vdash \lceil t_1 \rceil_\varepsilon : ?$, $\Xi; \Gamma \vdash \lceil t_2 \rceil_\varepsilon : ?$ and $\Xi; \Gamma, x : ? \vdash \lceil t_3 \rceil_\varepsilon : ?$. By the induction hypothesis, we know that $\mu; \Xi; \Gamma \vdash t_1 \approx \lceil t_1 \rceil_\varepsilon : ?$, $\mu; \Xi; \Gamma \vdash t_2 \approx \lceil t_2 \rceil_\varepsilon : ?$ and $\mu; \Xi; \Gamma, x : ? \vdash t_3 \approx \lceil t_3 \rceil_\varepsilon : ?$. Therefore, by the Rule (RunsL) the result follows immediately.

Case ($\langle t_1, t_2 \rangle$). Then, we know that

$$\lceil \langle t_1, t_2 \rangle \rceil_\varepsilon = \varepsilon_{?\times ?} \langle \lceil t_1 \rceil_\varepsilon, \lceil t_2 \rceil_\varepsilon \rangle :: ?$$

We have $t = \langle t_1, t_2 \rangle$. Then, we have to prove that $\mu; \Xi; \Gamma \vdash \langle t_1, t_2 \rangle \approx \varepsilon_{?\times ?} \langle \lceil t_1 \rceil_\varepsilon, \lceil t_2 \rceil_\varepsilon \rangle :: ? : ?$. Since $\Xi; \Gamma \vdash \lceil t \rceil_\varepsilon : ?$ and by Lemma 9.13, we know that $\Xi; \Gamma \vdash \lceil t_1 \rceil_\varepsilon : ?$ and $\Xi; \Gamma \vdash \lceil t_2 \rceil_\varepsilon : ?$. By the induction hypothesis, we know that $\mu; \Xi; \Gamma \vdash t_1 \approx \lceil t_1 \rceil_\varepsilon : ?$ and $\mu; \Xi; \Gamma \vdash t_2 \approx \lceil t_2 \rceil_\varepsilon : ?$. Therefore, by the Rule (Rpt) the result follows immediately.

Case ($op(\overline{t'})$). Then, we know that

$$\lceil op(\overline{t'}) \rceil_\varepsilon = \mathsf{let}\ \overline{x} : ? = \lceil \overline{t'} \rceil\ \mathsf{in}\ \varepsilon_B op(\varepsilon_{\overline{B}} \overline{x} :: \overline{B}) :: ?$$

We have $t = op(\overline{t'})$. Then, we have to prove that

$$\mu; \Xi; \Gamma \vdash op(\overline{t'}) \approx \mathsf{let}\ \overline{x} : ? = \lceil \overline{t'} \rceil\ \mathsf{in}\ \varepsilon_B op(\varepsilon_{\overline{B}} \overline{x} :: \overline{B}) :: ? : ?$$

Since $\Xi; \Gamma \vdash \lceil t \rceil_\varepsilon : ?$ and by Lemma 9.13, we know that $\Xi; \Gamma \vdash \overline{\lceil t' \rceil_\varepsilon} : ?$. By the induction hypothesis, we know that $\mu; \Xi; \Gamma \vdash \overline{t'} \approx \overline{\lceil t' \rceil_\varepsilon} : ?$. Therefore, by the Rule (Rop) the result follows immediately.

Case ($\nu x.t'$). Then, we know that

$$\lceil \nu x.t' \rceil_\varepsilon = \mathsf{let}\ x = su_\varepsilon\ \mathsf{in}\ \lceil t' \rceil_\varepsilon$$

We have $t = \nu x.t'$. Then, we have to prove that $\mu; \Xi; \Gamma \vdash \nu x.t' \approx \mathsf{let}\ x = su_\varepsilon\ \mathsf{in}\ \lceil t' \rceil_\varepsilon : ?$. Since $\Xi; \Gamma \vdash \mathsf{let}\ x = su_\varepsilon\ \mathsf{in}\ \lceil t' \rceil_\varepsilon : ?$, we know that $\Xi; \Gamma, x : ? \vdash \lceil t' \rceil_\varepsilon : ?$. By the induction hypothesis, we know that $\mu; \Xi; \Gamma, x : ? \vdash t' \approx \lceil t' \rceil_\varepsilon : ?$. Therefore, by the Rule (RsG) the result follows immediately.

□

Lemma 11.7. *If* $\vdash \lceil t \rceil \rightsquigarrow t_\varepsilon : ?$, *then* $\vdash t \approx t_\varepsilon : ?$.

Proof. Direct by 9.3. □

Lemma 9.4. *If* $\mu; \Xi \vdash v \approx t : ?$, *then* $\Xi \triangleright t \longmapsto^* \Xi \triangleright v'$, *and* $\mu; \Xi \vdash v \approx v' : ?$, *for some* v'.

Proof. The proof is a straightforward induction on the derivation of the rule $\mu; \Xi \vdash v \approx t : ?$. We only take into account rule cases where the term on the left can be a value.

Case (Rb). Trivial case because both terms in the relation are values.

$$(\mathrm{Rb})\frac{ty(b) = B}{\mu; \Xi \vdash b \approx \varepsilon_B b :: ? : ?}$$

Case (Rs). Trivial case because both terms in the relation are values.

$$(\mathrm{Rs})\frac{\sigma := ? \in \Xi}{\mu; \Xi \vdash \sigma \approx su^\sigma : ?}$$

Case (Ru).

$$(\mathrm{Ru})\frac{\mu; \Xi \vdash v \approx \varepsilon_D u :: ? : ?}{\mu; \Xi \vdash v \approx \varepsilon_D(\varepsilon_D u :: D) :: ? : ?}$$

If $t = \varepsilon_D(\varepsilon_D u :: D) :: ?$, then we know by the reduction rules of GSFε that:

$$\Xi \triangleright t \longmapsto \Xi \triangleright \varepsilon_D u :: ?$$

Note that $\varepsilon_D \,⨾\, \varepsilon_D = \varepsilon_D$ by Lemma 9.12. Then, we have to prove that $\mu; \Xi \vdash v \approx \varepsilon_D u :: ? : ?$, which is a premise. Therefore, the result follows immediately.

Case (Rp). Trivial case because both terms in the relation are values.

$$(\mathrm{Rp})\frac{\mu; \Xi \vdash v_1 \approx \varepsilon_{D_1} u_1 :: ? : ? \qquad \mu; \Xi \vdash v_2 \approx \varepsilon_{D_2} u_2 :: ? : ?}{\mu; \Xi \vdash \langle v_1, v_2 \rangle \approx \varepsilon_{D_1 \times D_2} \langle u_1, u_2 \rangle :: ? : ?}$$

Case (Rλ). Trivial case because both terms in the relation are values.

$$(\mathrm{R}\lambda)\frac{\mu; \Xi; x : ? \vdash t_1 \approx t_2 : ?}{\mu; \Xi \vdash (\lambda x.t_1) \approx \varepsilon_{? \to ?}(\lambda x.t_2) :: ? : ?}$$

Case (Rpt).

$$\text{(Rpt)}\frac{\mu;\Xi \vdash t_1 \approx t_1' : ? \qquad \mu;\Xi \vdash t_2 \approx t_2' : ?}{\mu;\Xi \vdash \langle t_1, t_2\rangle \approx \varepsilon_{?\times?}\langle t_1', t_2'\rangle :: ? : ?}$$

We have $t = \varepsilon_{?\times?}\langle t_1', t_2'\rangle :: ?$. We know that $\langle t_1, t_2\rangle = \langle v_1, v_2\rangle$ for some v_1 and v_2. Also, we know by premise that $\mu;\Xi \vdash v_1 \approx t_1' : ?$ and $\mu;\Xi \vdash v_2 \approx t_2' : ?$. Then, by the induction hypothesis, we know that exists v_1' and v_2' such that $\Xi \triangleright t_1' \longmapsto^* \Xi \triangleright v_1'$, $\Xi \triangleright t_2' \longmapsto^* \Xi \triangleright v_2'$, $\mu;\Xi \vdash v_1 \approx v_1' : ?$ and $\mu;\Xi \vdash v_2 \approx v_2' : ?$. Now, we have to prove that $\mu;\Xi \vdash \langle v_1, v_2\rangle \approx \varepsilon_{?\times?}\langle v_1', v_2'\rangle :: ? : ?$. But the result follows immediately by the rule (Rpt).

Case (Rsed1).

$$\text{(Rsed1)}\frac{\mu;\Xi \vdash t_1 \approx t_1' : ? \qquad \mu;\Xi \vdash t_2 \approx t_2' : ?}{\mu;\Xi \vdash \{t_1\}_{t_2} \approx \varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}t_1' :: ?\times ?) :: ? \to ?\ t_2' : ?}$$

We have $t = \varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}t_1' :: ?\times ?) :: ? \to ?\ t_2'$. Also, we know that $\{t_1\}_{t_2} = \{v\}_\sigma$, for some v and σ. Then, we know that $\mu;\Xi \vdash v \approx t_2' : ?$ and $\mu;\Xi \vdash \sigma \approx t_1' : ?$. Then, by the induction hypothesis, we know that exists v_1' and v_2' and such that $\Xi \triangleright t_1' \longmapsto^* \Xi \triangleright v_1'$, $\Xi \triangleright t_2' \longmapsto^* \Xi \triangleright v_2'$, $\mu;\Xi \vdash \sigma \approx v_1' : ?$ and $\mu;\Xi \vdash v \approx v_2' : ?$. By the rule (Rs), we know that $v_1' = su_\varepsilon^\sigma$. By the dynamic semantics of GSFε, we know that

$$\Xi \triangleright t \longmapsto^* \Xi \triangleright \varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}su_\varepsilon^\sigma :: ?\times ?) :: ? \to ?\ v_2' \longmapsto^*$$

$$\Xi \triangleright \langle \sigma^? \to ?, ? \to ?\rangle(\lambda x : \sigma.\varepsilon_{\sigma^?}x :: ?) :: ?\ v_2' \longmapsto \Xi \triangleright \varepsilon_{\sigma^?}(\langle E_1, \sigma^{E_2}\rangle u :: ?) :: ? \longmapsto \Xi \triangleright \langle E_1, \sigma^{E_2}\rangle u :: ?$$

where $v_2' = \langle E_1, E_2\rangle u :: ?$. Therefore, we have to prove that $\mu;\Xi \vdash \{v\}_\sigma \approx \langle E_1, \sigma^{E_2}\rangle u :: ? : ?$. As we know that $\mu;\Xi \vdash v \approx v_2' : ?$ or what is the same $\mu;\Xi \vdash v \approx \langle E_1, E_2\rangle u :: ? : ?$, by the Rule (Rsed2), the result follows immediately.

Case (Rsed1).

$$\text{(Rsed1)}\frac{\mu;\Xi;\Gamma \vdash v_1 \approx v_1' : ? \qquad \mu;\Xi;\Gamma \vdash v_2 \approx v_2' : ?}{\mu;\Xi;\Gamma \vdash \{v_1\}_{v_2} \approx \varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}v_2' :: ?\times ?) :: ? \to ?\ v_1' : ?}$$

We have $t = \varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}v_2' :: ?\times ?) :: ? \to ?\ v_1'$. Also, we know that $\{v_1\}_{v_2} = \{v\}_\sigma$, for some v and σ. Then, we know that $\mu;\Xi \vdash v \approx v_1' : ?$ and $\mu;\Xi \vdash \sigma \approx v_2' : ?$. By the rule (Rs), we know that $v_2' = su_\varepsilon^\sigma$. By the dynamic semantics of GSFε, we know that

$$\Xi \triangleright t \longmapsto^* \Xi \triangleright \varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}su_\varepsilon^\sigma :: ?\times ?) :: ? \to ?\ v_1' \longmapsto^*$$

$$\Xi \triangleright \langle \sigma^? \to ?, ? \to ?\rangle(\lambda x : \sigma.\varepsilon_{\sigma^?}x :: ?) :: ?\ v_1' \longmapsto \Xi \triangleright \varepsilon_{\sigma^?}(\langle E_1, \sigma^{E_2}\rangle u :: ?) :: ? \longmapsto \Xi \triangleright \langle E_1, \sigma^{E_2}\rangle u :: ?$$

where $v_1' = \langle E_1, E_2\rangle u :: ?$. Therefore, we have to prove that $\mu;\Xi \vdash \{v\}_\sigma \approx \langle E_1, \sigma^{E_2}\rangle u :: ? : ?$. As we know that $\mu;\Xi \vdash v \approx v_1' : ?$ or what is the same $\mu;\Xi \vdash v \approx \langle E_1, E_2\rangle u :: ? : ?$, by the Rule (Rsed2), the result follows immediately.

Case (Rsed1L).

$$\text{(Rsed1L)}\frac{\mu;\Xi;\Gamma \vdash t_1 \approx t_1' : ? \qquad \mu;\Xi;\Gamma \vdash t_2 \approx t_2' : ?}{\mu;\Xi;\Gamma \vdash \{t_1\}_{t_2} \approx \mathsf{let}\ x = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ?\times ?) :: ? \to ?)\ x : ?}$$

We have

$$t = \mathsf{let}\ x = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ?\times ?) :: ? \to ?)\ x$$

Also, we know that $\{t_1\}_{t_2} = \{v\}_\sigma$, for some v and σ. Then, we know that $\mu;\Xi \vdash v \approx t_1' : ?$ and $\mu;\Xi \vdash \sigma \approx t_2' : ?$. Then, by the induction hypothesis, we know that exists v_1' and v_2' such that $\Xi \triangleright t_1' \longmapsto^* \Xi \triangleright v_1'$, $\Xi \triangleright t_2' \longmapsto^* \Xi \triangleright v_2'$, $\mu;\Xi \vdash \sigma \approx v_2' : ?$ and $\mu;\Xi \vdash v \approx v_1' : ?$. By the rule (Rs), we know that $v_2' = su_\varepsilon^\sigma$. By the dynamic semantics of GSFε, we know that

$$\Xi \triangleright t \longmapsto^* \Xi \triangleright \varepsilon_?\varepsilon_?(\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}su_\varepsilon^\sigma :: ?\times ?) :: ? \to ?\ v_2') :: ? :: ? \longmapsto^*$$

$$\Xi \triangleright \varepsilon_?(\varepsilon_?\langle \sigma^? \to ?, ? \to ?\rangle(\lambda x : \sigma.\varepsilon_{\sigma^?} x :: ?) :: ?\ v_2') :: ? :: ? \longmapsto$$

$$\Xi \triangleright \varepsilon_?\varepsilon_?(\varepsilon_{\sigma^?}(\langle E_1, \sigma^{E_2}\rangle u :: ?) :: ?) :: ? :: ? \longmapsto \Xi \triangleright \langle E_1, \sigma^{E_2}\rangle u :: ?$$

where $v_1' = \langle E_1, E_2\rangle u :: ?$. Therefore, we have to prove that $\mu; \Xi \vdash \{v\}_\sigma \approx \langle E_1, \sigma^{E_2}\rangle u :: ? : ?$. As we know that $\mu; \Xi \vdash v \approx v_1' : ?$ or what is the same $\mu; \Xi \vdash v \approx \langle E_1, E_2\rangle u :: ? : ?$, by the Rule (Rsed2), the result follows immediately.

Case (Rsed1R).

$$\text{(Rsed1R)}\frac{\mu; \Xi; \Gamma \vdash v_1 \approx v_1' : ? \qquad \mu; \Xi; \Gamma \vdash t_2 \approx t_2' : ?}{\mu; \Xi; \Gamma \vdash \{v_1\}_{t_2} \approx \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{? \to ?}\pi_1(\varepsilon_{? \times ?} y :: ? \times ?) :: ? \to ?)\ v_1' : ?}$$

We have

$$t = \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{? \to ?}\pi_1(\varepsilon_{? \times ?} y :: ? \times ?) :: ? \to ?)\ v_1'$$

Also, we know that $\{v_1\}_{t_2} = \{v\}_\sigma$, for some v and σ. Then, we know that $\mu; \Xi \vdash v \approx v_1' : ?$ and $\mu; \Xi \vdash \sigma \approx t_2' : ?$. Then, by the induction hypothesis, we know that exists v_2' such that $\Xi \triangleright t_2' \longmapsto^* \Xi \triangleright v_2'$ and $\mu; \Xi \vdash \sigma \approx v_2' : ?$. By the rule (Rs), we know that $v_2' = su_\varepsilon^\sigma$. By the dynamic semantics of GSFε, we know that

$$\Xi \triangleright t \longmapsto^* \Xi \triangleright \varepsilon_?(\varepsilon_{? \to ?}\pi_1(\varepsilon_{? \times ?} su_\varepsilon^\sigma :: ? \times ?) :: ? \to ?\ v_2') :: ? \longmapsto^*$$

$$\Xi \triangleright \varepsilon_?(\langle \sigma^? \to ?, ? \to ?\rangle(\lambda x : \sigma.\varepsilon_{\sigma^?} x :: ?) :: ?\ v_2') :: ? \longmapsto$$

$$\Xi \triangleright \varepsilon_?(\varepsilon_{\sigma^?}(\langle E_1, \sigma^{E_2}\rangle u :: ?) :: ?) :: ? \longmapsto \Xi \triangleright \langle E_1, \sigma^{E_2}\rangle u :: ?$$

where $v_1' = \langle E_1, E_2\rangle u :: ?$. Therefore, we have to prove that $\mu; \Xi \vdash \{v\}_\sigma \approx \langle E_1, \sigma^{E_2}\rangle u :: ? : ?$. As we know that $\mu; \Xi \vdash v \approx v_1' : ?$ or what is the same $\mu; \Xi \vdash v \approx \langle E_1, E_2\rangle u :: ? : ?$, by the Rule (Rsed2), the result follows immediately.

Case (Rsed2). Trivial case because both terms in the relation are values.

$$\text{(Rsed2)}\frac{\mu; \Xi \vdash v \approx \langle E_1, E_2\rangle u :: ? : ? \qquad \sigma := ? \in \Xi}{\mu; \Xi \vdash \{v\}_\sigma \approx \langle E_1, \sigma^{E_2}\rangle u :: ? : ?}$$

Case (R?).

$$\text{(R?)}\frac{\mu; \Xi; \Gamma \vdash v \approx t' : ?}{\mu; \Xi; \Gamma \vdash v \approx \varepsilon_? t' :: ? : ?}$$

We have $t = \varepsilon_? t' :: ?$, where $\mu; \Xi; \Gamma \vdash v \approx t' : ?$. Then, by the induction hypothesis, we have that $\Xi \triangleright t' \longmapsto^* v''$ and $\mu; \Xi; \Gamma \vdash v \approx v'' : ?$. By the dynamic semantics of GSFε, we know that

$$\Xi \triangleright \varepsilon_? t' :: ? \longmapsto^* \Xi \triangleright \varepsilon_? v'' :: ? \longmapsto \Xi \triangleright v''$$

Therefore, the result follows immediately.

□

Lemma 11.3. *If $\mu; \Xi \vdash v \approx t_\varepsilon : ?$, then there exists v_ε s.t. $\Xi \triangleright t_\varepsilon \longmapsto^* \Xi \triangleright v_\varepsilon$, and $\mu; \Xi \vdash v \approx v_\varepsilon : ?$.*

Proof. Direct by Lemma 9.4. □

Lemma 9.5. *If $\mu; \Xi \vdash t \approx t_* : ?$ and $t \parallel \mu \longrightarrow t' \parallel \mu'$, then $\Xi \triangleright t_* \longmapsto^* \Xi' \triangleright t_*'$ and $\mu'; \Xi' \vdash t' \approx t_*' : ?$, for some t_*'.*

Proof. The proof is a straightforward induction on $\mu; \Xi \vdash t \approx t_* : ?$ and case analysis on $t \parallel \mu \longrightarrow t' \parallel \mu'$. The following rules are the only ones that can be applied in this case.

Case (RsG).

$$\text{(RsG)}\ \frac{\mu;\Xi;x:?\vdash t_1 \approx t_1' : ?}{\mu;\Xi \vdash \nu x.t_1 \approx \mathsf{let}\ x = su_\varepsilon\ \mathsf{in}\ t_1' : ?}$$

Since $t \parallel \mu \longrightarrow t' \parallel \mu'$, we know that $t = \nu x.t_1$. By the reduction rules of λ_{seal}, we know that $t \parallel \mu \longrightarrow t_1[\sigma/x] \parallel \mu, \sigma$. By Lemma 9.15, we know that $\Xi \triangleright su_\varepsilon \longmapsto^* \Xi, \sigma := ? \triangleright su_\varepsilon^\sigma$. By Rule (Rs), we know that $\mu, \sigma; \Xi, \sigma := ? \vdash \sigma \approx su_\varepsilon^\sigma : ?$. By the reduction rules of GSFε, we know that

$$\Xi \triangleright \mathsf{let}\ x = su_\varepsilon\ \mathsf{in}\ t_1' \longmapsto^* \Xi, \sigma := ? \triangleright \mathsf{let}\ x = su_\varepsilon^\sigma\ \mathsf{in}\ t_1' \longmapsto \Xi, \sigma := ? \triangleright \varepsilon_?(t_1'[su_\varepsilon^\sigma/x]) :: ?$$

Then, we are required to show that $\mu, \sigma; \Xi, \sigma := ? \vdash t_1[\sigma/x] \approx \varepsilon_?(t_1'[su_\varepsilon^\sigma/x]) :: ? : ?$. We know by the premise that $\mu; \Xi; x : ? \vdash t_1 \approx t_1' : ?$, or what is the same $\mu, \sigma; \Xi, \sigma := ?; x : ? \vdash t_1 \approx t_1' : ?$. Since $\mu, \sigma; \Xi, \sigma := ?; x : ? \vdash t_1 \approx t_1' : ?$ and $\mu, \sigma; \Xi, \sigma := ? \vdash \sigma \approx su_\varepsilon^\sigma : ?$, by the Lemma 9.16 and Rule (R?) the result follows immediately.

Case (Runs).

$$\text{(Runs)}\ \frac{\mu;\Xi \vdash v_1 \approx v_1' : ? \qquad \mu;\Xi \vdash v_2 \approx v_2' : ? \qquad \mu;\Xi;z:?\vdash t_3 \approx t_3' : ?}{\mu;\Xi \vdash \mathsf{let}\ \{z\}_{v_1} = v_2\ \mathsf{in}\ t_3 \approx \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}v_1' :: ?\times?) :: ? \to ?\ v_2'\ \mathsf{in}\ t_3' : ?}$$

Since $t \parallel \mu \longrightarrow t' \parallel \mu'$, we know that $t = \mathsf{let}\ \{z\}_\sigma = \{v\}_\sigma\ \mathsf{in}\ t_3$. By the reduction rules of λ_{seal}, we know that $t \parallel \mu \longrightarrow t_3[v/z] \parallel \mu$. We know by the premises that $\mu; \Xi \vdash \sigma \approx v_1' : ?$ and $\mu; \Xi \vdash \{v\}_\sigma \approx v_2' : ?$. Therefore, by Rules (Rs) and (Rsed2), we know that $v_1 = su_\varepsilon^\sigma$ and $v_2 = \langle E_1, \sigma^{E_2}\rangle u :: ?$, for some u, E_1 and E_2. By the reduction rules of GSFε, we know that

$$\Xi \triangleright \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}su_\varepsilon^\sigma :: ?\times?) :: ? \to ?\ (\langle E_1, \sigma^{E_2}\rangle u :: ?)\ \mathsf{in}\ t_3' \longmapsto^*$$
$$\Xi \triangleright \mathsf{let}\ z = (\langle ? \to \sigma^?, ? \to ?\rangle(\lambda x : ?.\varepsilon_{\sigma^?} x :: \sigma) :: ?)\ (\langle E_1, \sigma^{E_2}\rangle u :: ?)\ \mathsf{in}\ t_3' \longmapsto^*$$
$$\Xi \triangleright \mathsf{let}\ z = (\langle E_1, E_2\rangle u :: ?)\ \mathsf{in}\ t_3' \longmapsto^* \Xi \triangleright t_3'[\langle E_1, E_2\rangle u :: ?/x]$$

We are required to show that $\mu; \Xi \vdash t_3[v/z] \approx t_3'[\langle E_1, E_2\rangle u :: ?/z] : ?$, but we know that $\mu; \Xi \vdash \{v\}_\sigma \approx \langle E_1, \sigma^{E_2}\rangle u :: ? : ?$, therefore we know by the rule (Rsed2) that $\mu; \Xi \vdash v \approx \langle E_1, E_2\rangle u :: ? : ?$. Finally, by the Lemma 9.16, the result follows immediately.

Case (Rop).

$$\text{(Rop)}\ \frac{\mu;\Xi;\Gamma \vdash \overline{t_1} \approx \overline{t_2} : \overline{B} \qquad ty(op) = \overline{B} \to B'}{\mu;\Xi;\Gamma \vdash op(\overline{t_1}) \approx op(\varepsilon_{\overline{B}}\overline{t_2} :: \overline{B}) :: ? : B'}$$

Applying the induction hypothesis, reduction rules of λ_{seal} and GSFε, and Rule (Rδ).

Case (RunsL).

$$\text{(RunsL)}\ \frac{\mu;\Xi \vdash t_1 \approx t_1' : ? \qquad \mu;\Xi \vdash t_2 \approx t_2' : ? \qquad \mu;\Xi;z:?\vdash t_3 \approx t_3' : ?}{\mu;\Xi \vdash \mathsf{let}\ \{z\}_{t_1} = t_2\ \mathsf{in}\ t_3 \approx \mathsf{let}\ x = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}x :: ?\times?) :: ? \to ?\ y\ \mathsf{in}\ t_3' : ?}$$

Since $t \parallel \mu \longrightarrow t' \parallel \mu'$, we know that $t = \mathsf{let}\ \{z\}_\sigma = \{v\}_\sigma\ \mathsf{in}\ t_3$. By the reduction rules of λ_{seal}, we know that $t \parallel \mu \longrightarrow t_3[v/z] \parallel \mu$. We know by the premises that $\mu; \Xi \vdash \sigma \approx t_1' : ?$ and $\mu; \Xi \vdash \{v\}_\sigma \approx t_2' : ?$. Therefore, by Lemma 9.4, we know that $\Xi \triangleright t_1' \longmapsto^* \Xi_1 \triangleright v_1$, $\Xi \triangleright t_2' \longmapsto^* \Xi \triangleright v_2$, $\mu; \Xi \vdash \sigma \approx v_1 : ?$ and $\mu; \Xi \vdash \{v\}_\sigma \approx v_2 : ?$, for some v_1 and v_2. By Rules (Rs) and (Rsed2), we know that $v_1 = su_\varepsilon^\sigma$ and $v_2 = \langle E_1, \sigma^{E_2}\rangle u :: ?$, for some u, E_1 and E_2. By the reduction rules of GSFε, we know that

$$\Xi \triangleright \mathsf{let}\ x = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}x :: ?\times?) :: ? \to ?\ y\ \mathsf{in}\ t_3' \longmapsto^*$$
$$\Xi \triangleright \varepsilon_?\varepsilon_?(\mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}su_\varepsilon^\sigma :: ?\times?) :: ? \to ?\ (\langle E_1, \sigma^{E_2}\rangle u :: ?)\ \mathsf{in}\ t_3') :: ? :: ? \longmapsto^*$$
$$\Xi \triangleright \varepsilon_?\varepsilon_?(\mathsf{let}\ z = (\langle ? \to \sigma^?, ? \to ?\rangle(\lambda x : ?.\varepsilon_{\sigma^?} x :: \sigma) :: ?)\ (\langle E_1, \sigma^{E_2}\rangle u :: ?)\ \mathsf{in}\ t_3') :: ? :: ? \longmapsto^*$$

$$\Xi \triangleright \varepsilon_? \varepsilon_? (\mathsf{let}\ z = (\langle E_1, E_2 \rangle u :: ?)\ \mathsf{in}\ t_3') :: ? :: ? \longmapsto^* \Xi \triangleright \varepsilon_? \varepsilon_? \varepsilon_? (t_3'[\langle E_1, E_2 \rangle u :: ?/x]) :: ? :: ? :: ?$$

We are required to show that $\mu; \Xi \vdash t_3[v/z] \approx \varepsilon_? \varepsilon_? \varepsilon_? (t_3'[\langle E_1, E_2 \rangle u :: ?/z]) :: ? :: ? :: ? : ?$, but we know that $\mu; \Xi \vdash \{v\}_\sigma \approx \langle E_1, \sigma^{E_2} \rangle u :: ? : ?$, therefore we know by the rule (Rsed2) that $\mu; \Xi \vdash v \approx \langle E_1, E_2 \rangle u :: ? : ?$. Finally, by the Lemma 9.16 and the Rule (R?), the result follows immediately.

Case (RunsR).

$$\text{(RunsR)}\ \frac{\mu; \Xi \vdash v_1 \approx v_1' : ? \qquad \mu; \Xi \vdash t_2 \approx t_2' : ? \qquad \mu; \Xi; z : ? \vdash t_3 \approx t_3' : ?}{\mu; \Xi \vdash \mathsf{let}\ \{z\}_{v_1} = t_2\ \mathsf{in}\ t_3 \approx \mathsf{let}\ y = t_2'\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{? \to ?} \pi_2 (\varepsilon_{? \times ?} v_1' :: ? \times ?) :: ? \to ?\ y\ \mathsf{in}\ t_3' : ?}$$

Since $t \parallel \mu \longrightarrow t' \parallel \mu'$, we know that $t = \mathsf{let}\ \{z\}_\sigma = \{v\}_\sigma\ \mathsf{in}\ t_3$. By the reduction rules of λ_{seal}, we know that $t \parallel \mu \longrightarrow t_3[v/z] \parallel \mu$. We know by the premises that $\mu; \Xi \vdash \sigma \approx v_1' : ?$ and $\mu; \Xi \vdash \{v\}_\sigma \approx t_2' : ?$. Therefore, by Lemma 9.4, we know that $\Xi \triangleright t_2' \longmapsto^* \Xi \triangleright v_2$ and $\mu; \Xi \vdash \{v\}_\sigma \approx v_2 : ?$, for some v_2. By Rules (Rs) and (Rsed2), we know that $v_1 = su_\varepsilon^\sigma$ and $v_2 = \langle E_1, \sigma^{E_2} \rangle u :: ?$, for some u, E_1 and E_2. By the reduction rules of GSFε, we know that

$$\Xi \triangleright \mathsf{let}\ y = t_2'\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{? \to ?} \pi_2 (\varepsilon_{? \times ?} v_1' :: ? \times ?) :: ? \to ?\ y\ \mathsf{in}\ t_3' \longmapsto^*$$

$$\Xi \triangleright \varepsilon_? (\mathsf{let}\ z = \varepsilon_{? \to ?} \pi_2 (\varepsilon_{? \times ?} su_\varepsilon^\sigma :: ? \times ?) :: ? \to ?\ (\langle E_1, \sigma^{E_2} \rangle u :: ?)\ \mathsf{in}\ t_3') :: ? \longmapsto^*$$

$$\Xi \triangleright \varepsilon_? (\mathsf{let}\ z = (\langle ? \to \sigma^?, ? \to ? \rangle (\lambda x : ?. \varepsilon_{\sigma^?} x :: \sigma) :: ?)\ (\langle E_1, \sigma^{E_2} \rangle u :: ?)\ \mathsf{in}\ t_3') :: ? \longmapsto^*$$

$$\Xi \triangleright \varepsilon_? (\mathsf{let}\ z = (\langle E_1, E_2 \rangle u :: ?)\ \mathsf{in}\ t_3') :: ? \longmapsto \Xi \triangleright \varepsilon_? \varepsilon_? (t_3'[\langle E_1, E_2 \rangle u :: ?/x]) :: ? :: ?$$

We are required to show that $\mu; \Xi \vdash t_3[v/z] \approx \varepsilon_? \varepsilon_? (t_3'[\langle E_1, E_2 \rangle u :: ?/z]) :: ? :: ? : ?$, but we know that $\mu; \Xi \vdash \{v\}_\sigma \approx \langle E_1, \sigma^{E_2} \rangle u :: ? : ?$, therefore we know by the rule (Rsed2) that $\mu; \Xi \vdash v \approx \langle E_1, E_2 \rangle u :: ? : ?$. Finally, by the Lemma 9.16 and the Rule (R?), the result follows immediately.

Case (Rapp).

$$\text{(Rapp)}\ \frac{\mu; \Xi \vdash v_1 \approx v_1' : ? \qquad \mu; \Xi \vdash v_2 \approx v_2' : ?}{\mu; \Xi \vdash v_1\ v_2 \approx (\varepsilon_{? \to ?} v_1' :: ? \to ?)\ v_2' : ?}$$

Since $t \parallel \mu \longrightarrow t' \parallel \mu'$, we know that $t = (\lambda x. t_1'')\ v_2$, where $v_1 = (\lambda x. t_1'')$. Therefore, we know that $\mu; \Xi \vdash (\lambda x. t_1'') \approx v_1' : ?$ and $\mu; \Xi \vdash v_2 \approx v_2' : ?$. By the rule (R$\lambda$), we know that $v_1' = \varepsilon_{? \to ?} \lambda x. t_1''' :: ?$, where $\Xi; x : ? \vdash t_1'' \approx t_1''' : ?$.

By the dynamic semantics of λ_{seal}, we know that

$$(\lambda x. t_1'')\ v_2 \parallel \mu \longrightarrow t_1''[v_2/x] \parallel \mu$$

By the dynamic semantics of GSFε, we know that

$$\Xi \triangleright (\varepsilon_{? \to ?} (\varepsilon_{? \to ?} \lambda x. t_1''' :: ?) :: ? \to ?)\ v_2' \longmapsto^*$$

$$\Xi \triangleright (\varepsilon_{? \to ?} (\lambda x. t_1''') :: ? \to ?)\ v_2' \longmapsto \Xi \triangleright \varepsilon_? (t_1'''[v_2'/x]) :: ?$$

Since $\mu; \Xi; x : ? \vdash t_1'' \approx t_1''' : ?$ and $\mu; \Xi \vdash v_2 \approx v_2' : ?$, we know by Lemma 9.16 that $\mu; \Xi \vdash t_1''[v_2/x] \approx (t_1'''[v_2'/x]) : ?$. By the Rule (R?), we know that $\mu; \Xi \vdash t_1''[v_2/x] \approx \varepsilon_? (t_1'''[v_2'/x]) :: ? : ?$, thus the result follows.

Case (RappL).

$$\text{(RappL)}\ \frac{\mu; \Xi \vdash t_1 \approx t_1' : ? \qquad \mu; \Xi \vdash t_2 \approx t_2' : ?}{\mu; \Xi \vdash t_1\ t_2 \approx \mathsf{let}\ x = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{? \to ?} x :: ? \to ?)\ y : ?}$$

Since $t \parallel \mu \longrightarrow t' \parallel \mu'$, we know that $t = (\lambda x. t_1'')\ v_2$, where $t_1 = (\lambda x. t_1'')$ and $t_2 = v_2$. Therefore, we know that $\mu; \Xi \vdash (\lambda x. t_1'') \approx t_1' : ?$ and $\mu; \Xi \vdash v_2 \approx t_2' : ?$. By Lemma 9.4, we know that

$\Xi \triangleright t_1' \longmapsto^* \Xi \triangleright v_1'$, $\Xi \triangleright t_2' \longmapsto^* \Xi \triangleright v_2'$, $\mu;\Xi \vdash (\lambda x.t_1'') \approx v_1' : ?$ and $\mu;\Xi \vdash v_2 \approx v_2' : ?$, for some v_1' and v_2'. By the rule (Rλ), we know that $v_1' = \varepsilon_{?\to?}\lambda x.t_1''' :: ?$, where $\Xi; x : ? \vdash t_1'' \approx t_1''' : ?$.

By the dynamic semantics of λ_{seal}, we know that

$$(\lambda x.t_1'')\ v_2 \parallel \mu \longrightarrow t_1''[v_2/x] \parallel \mu$$

By the dynamic semantics of GSFε, we know that

$$\Xi \triangleright \mathsf{let}\ x = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?}x :: ? \to ?)\ y \longmapsto^* \Xi \triangleright \varepsilon_?\varepsilon_?\varepsilon_{?\to?}(\lambda x.t_1''') :: ? \to ?)\ v_2') :: ? :: ?$$
$$\Xi \triangleright \varepsilon_?\varepsilon_?(\varepsilon_?(t_1'''[v_2'/x]) :: ?) :: ? :: ?$$

Since $\mu;\Xi;x : ? \vdash t_1'' \approx t_1''' : ?$ and $\mu;\Xi \vdash v_2 \approx v_2' : ?$, we know by Lemma 9.16 that $\mu;\Xi \vdash t_1''[v_2/x] \approx (t_1'''[v_2'/x]) : ?$. By the Rule (R?), we know that $\mu;\Xi \vdash t_1''[v_2/x] \approx \varepsilon_?(t_1'''[v_2'/x]) :: ? : ?$, therefore we have $\mu;\Xi \vdash t_1''[v_2/x] \approx \varepsilon_?\varepsilon_?(\varepsilon_?(t_1'''[v_2'/x]) :: ?) :: ? :: ? : ?$, thus the result follows.

Case (RappR).

$$\text{(RappR)}\frac{\mu;\Xi \vdash v_1 \approx v_1' : ? \qquad \mu;\Xi \vdash t_2 \approx t_2' : ?}{\mu;\Xi \vdash v_1\ t_2 \approx \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?}v_1' :: ? \to ?)\ y : ?}$$

Since $t \parallel \mu \longrightarrow t' \parallel \mu'$, we know that $t = (\lambda x.t_1'')\ v_2$, where $v_1 = (\lambda x.t_1'')$ and $t_2 = v_2$. Therefore, we know that $\mu;\Xi \vdash (\lambda x.t_1'') \approx v_1' : ?$ and $\mu;\Xi \vdash v_2 \approx t_2' : ?$. By Lemma 9.4, we know that $\Xi \triangleright t_2' \longmapsto^* \Xi \triangleright v_2'$ and $\mu;\Xi \vdash v_2 \approx v_2' : ?$, for some v_2'. By the rule (Rλ), we know that $v_1' = \varepsilon_{?\to?}\lambda x.t_1''' :: ?$, where $\Xi; x : ? \vdash t_1'' \approx t_1''' : ?$.

By the dynamic semantics of λ_{seal}, we know that

$$(\lambda x.t_1'')\ v_2 \parallel \mu \longrightarrow t_1''[v_2/x] \parallel \mu$$

By the dynamic semantics of GSFε, we know that

$$\Xi \triangleright \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?}v_1' :: ? \to ?)\ y \longmapsto^* \Xi \triangleright \varepsilon_?\varepsilon_{?\to?}(\lambda x.t_1''') :: ? \to ?)\ v_2') :: ?$$
$$\Xi \triangleright \varepsilon_?(\varepsilon_?(t_1'''[v_2'/x]) :: ?) :: ?$$

Since $\mu;\Xi;x : ? \vdash t_1'' \approx t_1''' : ?$ and $\mu;\Xi \vdash v_2 \approx v_2' : ?$, we know by Lemma 9.16 that $\mu;\Xi \vdash t_1''[v_2/x] \approx (t_1'''[v_2'/x]) : ?$. By the Rule (R?), we know that $\mu;\Xi \vdash t_1''[v_2/x] \approx \varepsilon_?(t_1'''[v_2'/x]) :: ? : ?$, therefore we have $\mu;\Xi \vdash t_1''[v_2/x] \approx \varepsilon_?(\varepsilon_?(t_1'''[v_2'/x]) :: ?) :: ? : ?$, thus the result follows.

Case (Rpi).

$$\text{(Rpi)}\frac{\mu;\Xi \vdash t \approx t' : ?}{\mu;\Xi \vdash \pi_i(t) \approx \pi_i(\varepsilon_{?\times?}t' :: ? \times ?) : ?}$$

Applying the induction hypothesis, reduction rules of λ_{seal} and GSFε, and Rules (Rp) and (Rpt).

Case (R?). We have that

$$\text{(R?)}\frac{\mu;\Xi \vdash t \approx t_*'' : ?}{\mu;\Xi \vdash t \approx \varepsilon_? t_*'' :: ? : ?}$$

We have $t_* = \varepsilon_? t_*'' :: ?$, where $\mu;\Xi \vdash t \approx t_*'' : ?$. Then, by the induction hypothesis, we have that $\Xi \triangleright t_*'' \longmapsto^* \Xi' \triangleright t_*'''$ and $\mu';\Xi' \vdash t' \approx t_*''' : ?$. We are required to show that $\mu';\Xi' \vdash t' \approx \varepsilon_? t_*''' :: ? : ?$. But the result follows immediately by the Rule (R?).

□

Lemma 9.6. *Let $\mu;\Xi \vdash v_1 \approx \varepsilon u :: ? : ?$. Then, $v_1 = \lambda x.t_1$ if and only if $u = \lambda x : ?.t_2$ and $\varepsilon = \varepsilon_{?\to?}$.*

Proof. The proof follow by the exploration of rules in $\mu;\Xi \vdash v_1 \approx \varepsilon u :: ? : ?$ and the definition of the evidence. □

Corollary 9.7. *Let* $\mu;\Xi \vdash v_1 \approx \varepsilon u :: ? : ?$. *Then,* $v_1 \neq \lambda x.t_1$ *then* $u \neq \lambda x : ?.t_2$ *and* $\varepsilon \neq \varepsilon_{G_1 \to G_2}$.

Proof. By Lemma 9.6. □

Lemma 9.8. *If* $\mu;\Xi \vdash t \approx t_* : ?$ *and* $t \parallel \mu \longrightarrow$ **error**, *then* $\Xi \triangleright t_* \longmapsto^* \textbf{error}$.

Proof. The proof is a straightforward induction on $\mu;\Xi \vdash t \approx t_* : ?$. The following rule is the only one that can be applied in this case ($t \parallel \mu \longrightarrow$ **error**).

Case (Rapp).

$$\text{(Rapp)}\frac{\mu;\Xi \vdash v_1 \approx v_1' : ? \qquad \mu;\Xi \vdash v_2 \approx v_2' : ?}{\mu;\Xi \vdash v_1\ v_2 \approx (\varepsilon_{?\to?} v_1' :: ? \to ?)\ v_2' : ?}$$

Since $t \parallel \mu \longrightarrow$ **type_error**, we know that v_1 is not a function, and by Corollary 9.7 and $\mu;\Xi \vdash v_1 \approx v_1' : ?$, we know that v_1' also can not be a function and its evidence, syntactically, can not be a function. Let suppose that $v_1' = \varepsilon_1 u_1 :: ?$. Then, we know that $\varepsilon_1 \mathbin{⨟} \varepsilon_{?\to?}$ fails, and the result holds.

$$\Xi \triangleright (\varepsilon_{?\to?}(\varepsilon_1 u_1 :: ?) :: ? \to ?)\ v_2' \longmapsto^* \textbf{error}$$

Case (RappL).

$$\text{(RappL)}\frac{\mu;\Xi \vdash t_1 \approx t_1' : ? \qquad \mu;\Xi \vdash t_2 \approx t_2' : ?}{\mu;\Xi \vdash t_1\ t_2 \approx \mathsf{let}\ x = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?} x :: ? \to ?)\ y : ?}$$

By Lemma 9.4, $\mu;\Xi \vdash t_1 \approx t_1' : ?$ and $\mu;\Xi \vdash t_2 \approx t_2' : ?$, we know that $\Xi \triangleright t_1' \longmapsto^* \Xi \triangleright v_1'$, $\Xi \triangleright t_2' \longmapsto^* \Xi \triangleright v_2'$, $\mu;\Xi \vdash v_1 \approx v_1' : ?$ and $\mu;\Xi \vdash v_2 \approx v_2' : ?$, for some v_1' and v_2'. Since $t \parallel \mu \longrightarrow$ **type_error**, we know that v_1 is not a function, and by Corollary 9.7 and $\mu;\Xi \vdash v_1 \approx v_1' : ?$, we know that v_1' also can not be a function and its evidence, syntactically, can not be a function. Let suppose that $v_1' = \varepsilon_1 u_1 :: ?$. Then, we know that $\varepsilon_1 \mathbin{⨟} \varepsilon_{?\to?}$ fails, and the result holds.

$$\begin{gathered}\Xi \triangleright \mathsf{let}\ x = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?} x :: ? \to ?)\ y \longmapsto^* \\ \Xi \triangleright \varepsilon_? \varepsilon_? (\varepsilon_{?\to?}(\varepsilon_1 u_1 :: ?) :: ? \to ?)\ v_2') :: ? :: ? \longmapsto \textbf{error}\end{gathered}$$

Case (RappR).

$$\text{(RappR)}\frac{\mu;\Xi \vdash v_1 \approx v_1' : ? \qquad \mu;\Xi \vdash t_2 \approx t_2' : ?}{\mu;\Xi \vdash v_1\ t_2 \approx \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?} v_1' :: ? \to ?)\ y : ?}$$

By Lemma 9.4 and $\mu;\Xi \vdash t_2 \approx t_2' : ?$, we know that $\Xi \triangleright t_2' \longmapsto^* \Xi \triangleright v_2'$ and $\mu;\Xi \vdash v_2 \approx v_2' : ?$, for some v_2'. Since $t \parallel \mu \longrightarrow$ **type_error**, we know that v_1 is not a function, and by Corollary 9.7 and $\mu;\Xi \vdash v_1 \approx v_1' : ?$, we know that v_1' also can not be a function and its evidence, syntactically, can not be a function. Let suppose that $v_1' = \varepsilon_1 u_1 :: ?$. Then, we know that $\varepsilon_1 \mathbin{⨟} \varepsilon_{?\to?}$ fails, and the result holds.

$$\begin{gathered}\Xi \triangleright \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?} v_1' :: ? \to ?)\ y \longmapsto^* \\ \Xi \triangleright \varepsilon_? \varepsilon_{?\to?}(\varepsilon_1 u_1 :: ?) :: ? \to ?)\ v_2') :: ? \longmapsto \textbf{error}\end{gathered}$$

Case (TRpi). $\text{(TRpi)}\dfrac{\mu;\Xi;\Gamma \vdash t \approx t' : ?}{\mu;\Xi;\Gamma \vdash \pi_i(t) \approx \pi_i(\varepsilon_{?\times?} t' :: ? \times ?) : ?}$ Similar to the function application case.

Case (Runs).

$$\text{(Runs)}\frac{\mu;\Xi \vdash v_1 \approx v_1' : ? \qquad \mu;\Xi \vdash v_2 \approx v_2' : ? \qquad \Xi; z : ? \vdash t_3 \approx t_3' : ?}{\mu;\Xi \vdash \mathsf{let}\ \{z\}_{v_1} = v_2\ \mathsf{in}\ t_3 \approx \mathsf{let}\ z = \varepsilon_{?\to?} \pi_2(\varepsilon_{?\times?} v_1' :: ? \times ?) :: ? \to ?\ v_2'\ \mathsf{in}\ t_3' : ?}$$

Since $t \parallel \mu \longrightarrow$ **unseal_error**, we know that $t = \mathsf{let}\ \{z\}_\sigma = \{v\}_{\sigma'}\ \mathsf{in}\ t_3$, where $\sigma \neq \sigma'$. We know by the premises that $\mu; \Xi \vdash \sigma \approx v_1' : ?$ and $\mu; \Xi \vdash \{v\}_{\sigma'} \approx v_2' : ?$.Therefore, by Rules (Rs) and (Rsed2), we know that $v_1 = su_\varepsilon^\sigma$ and $v_2 = \langle E_1, \sigma'^{E_2}\rangle u :: ?$, for some u, E_1 and E_2. By the reduction rules of GSFε, we know that

$$\Xi \triangleright \mathsf{let}\ x = \varepsilon_{? \to ?}\pi_2(\varepsilon_{? \times ?} su_\varepsilon^\sigma :: ? \times ?) :: ? \to ?\ (\langle E_1, \sigma'^{E_2}\rangle u :: ?)\ \mathsf{in}\ t_3' \longmapsto^*$$

$$\Xi \triangleright \mathsf{let}\ x = (\langle ? \to \sigma^?, ? \to ?\rangle(\lambda x : ?.\varepsilon_{\sigma^?} x :: \sigma) :: ?)\ (\langle E_1, \sigma'^{E_2}\rangle u :: ?)\ \mathsf{in}\ t_3' \longmapsto^*$$

$$\Xi \triangleright \mathsf{let}\ x = (\langle \sigma^?, ?\rangle(\varepsilon_{\sigma^?}(\langle E_1, \sigma'^{E_2}\rangle u :: ?) :: \sigma) :: ?)\ \mathsf{in}\ t_3' \longmapsto \mathbf{error}$$

Note that the transitivity between $\langle E_1, \sigma'^{E_2}\rangle \circ \varepsilon_{\sigma^?}$ fails because $\sigma' \neq \sigma$. Thus the results follows immediately.

Case (RunsL).

$$\text{(RunsL)}\ \frac{\mu; \Xi \vdash t_1 \approx t_1' : ? \qquad \mu; \Xi \vdash t_2 \approx t_2' : ? \qquad \mu; \Xi; z : ? \vdash t_3 \approx t_3' : ?}{\mu; \Xi \vdash \mathsf{let}\ \{z\}_{t_1} = t_2\ \mathsf{in}\ t_3 \approx \mathsf{let}\ x = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{? \to ?}\pi_2(\varepsilon_{? \times ?} x :: ? \times ?) :: ? \to ?\ y\ \mathsf{in}\ t_3' : ?}$$

Since $t \parallel \mu \longrightarrow$ **unseal_error**, we know that $t = \mathsf{let}\ \{z\}_\sigma = \{v\}_{\sigma'}\ \mathsf{in}\ t_3$, where $\sigma \neq \sigma'$. We know by the premises that $\mu; \Xi \vdash \sigma \approx t_1' : ?$ and $\mu; \Xi \vdash \{v\}_{\sigma'} \approx t_2' : ?$. Therefore, by Lemma 9.4, we know that $\Xi \triangleright t_1' \longmapsto^* \Xi_1 \triangleright v_1$, $\Xi \triangleright t_2' \longmapsto^* \Xi \triangleright v_2$, $\mu; \Xi \vdash \sigma \approx v_1 : ?$ and $\mu; \Xi \vdash \{v\}_{\sigma'} \approx v_2 : ?$, for some v_1 and v_2. By Rules (Rs) and (Rsed2), we know that $v_1 = su_\varepsilon^\sigma$ and $v_2 = \langle E_1, \sigma'^{E_2}\rangle u :: ?$, for some u, E_1 and E_2. By the reduction rules of GSFε, we know that

$$\Xi \triangleright \mathsf{let}\ x = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{? \to ?}\pi_2(\varepsilon_{? \times ?} x :: ? \times ?) :: ? \to ?\ y\ \mathsf{in}\ t_3' \longmapsto^*$$

$$\Xi \triangleright \varepsilon_? \varepsilon_? (\mathsf{let}\ z = \varepsilon_{? \to ?}\pi_2(\varepsilon_{? \times ?} su_\varepsilon^\sigma :: ? \times ?) :: ? \to ?\ (\langle E_1, \sigma'^{E_2}\rangle u :: ?)\ \mathsf{in}\ t_3') :: ? :: ? \longmapsto^*$$

$$\Xi \triangleright \varepsilon_? \varepsilon_? (\mathsf{let}\ z = (\langle ? \to \sigma^?, ? \to ?\rangle(\lambda x : ?.\varepsilon_{\sigma^?} x :: \sigma) :: ?)\ (\langle E_1, \sigma'^{E_2}\rangle u :: ?)\ \mathsf{in}\ t_3') :: ? :: ? \longmapsto^*$$

$$\Xi \triangleright \varepsilon_? \varepsilon_? (\mathsf{let}\ x = (\langle \sigma^?, ?\rangle(\varepsilon_{\sigma^?}(\langle E_1, \sigma'^{E_2}\rangle u :: ?) :: \sigma) :: ?)\ \mathsf{in}\ t_3') :: ? :: ? \longmapsto \mathbf{error}$$

Note that the transitivity between $\langle E_1, \sigma'^{E_2}\rangle \circ \varepsilon_{\sigma^?}$ fails because $\sigma' \neq \sigma$. Thus the results follows immediately.

Case (RunsR).

$$\text{(RunsR)}\ \frac{\mu; \Xi \vdash v_1 \approx v_1' : ? \qquad \mu; \Xi \vdash t_2 \approx t_2' : ? \qquad \mu; \Xi; z : ? \vdash t_3 \approx t_3' : ?}{\mu; \Xi \vdash \mathsf{let}\ \{z\}_{v_1} = t_2\ \mathsf{in}\ t_3 \approx \mathsf{let}\ y = t_2'\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{? \to ?}\pi_2(\varepsilon_{? \times ?} v_1' :: ? \times ?) :: ? \to ?\ y\ \mathsf{in}\ t_3' : ?}$$

Since $t \parallel \mu \longrightarrow$ **unseal_error**, we know that $t = \mathsf{let}\ \{z\}_\sigma = \{v\}_{\sigma'}\ \mathsf{in}\ t_3$, where $\sigma \neq \sigma'$. We know by the premises that $\mu; \Xi \vdash \sigma \approx v_1' : ?$ and $\mu; \Xi \vdash \{v\}_{\sigma'} \approx t_2' : ?$. Therefore, by Lemma 9.4, we know that $\Xi \triangleright t_2' \longmapsto^* \Xi \triangleright v_2$ and $\mu; \Xi \vdash \{v\}_{\sigma'} \approx v_2 : ?$, for some v_2. By Rules (Rs) and (Rsed2), we know that $v_1 = su_\varepsilon^\sigma$ and $v_2 = \langle E_1, \sigma'^{E_2}\rangle u :: ?$, for some u, E_1 and E_2. By the reduction rules of GSFε, we know that

$$\Xi \triangleright \mathsf{let}\ y = t_2'\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{? \to ?}\pi_2(\varepsilon_{? \times ?} v_1' :: ? \times ?) :: ? \to ?\ y\ \mathsf{in}\ t_3' \longmapsto^*$$

$$\Xi \triangleright \varepsilon_? (\mathsf{let}\ z = \varepsilon_{? \to ?}\pi_2(\varepsilon_{? \times ?} su_\varepsilon^\sigma :: ? \times ?) :: ? \to ?\ (\langle E_1, \sigma'^{E_2}\rangle u :: ?)\ \mathsf{in}\ t_3') :: ? \longmapsto^*$$

$$\Xi \triangleright \varepsilon_? (\mathsf{let}\ z = (\langle ? \to \sigma^?, ? \to ?\rangle(\lambda x : ?.\varepsilon_{\sigma^?} x :: \sigma) :: ?)\ (\langle E_1, \sigma'^{E_2}\rangle u :: ?)\ \mathsf{in}\ t_3') :: ? \longmapsto^*$$

$$\Xi \triangleright \varepsilon_? (\mathsf{let}\ x = (\langle \sigma^?, ?\rangle(\varepsilon_{\sigma^?}(\langle E_1, \sigma'^{E_2}\rangle u :: ?) :: \sigma) :: ?)\ \mathsf{in}\ t_3') :: ? \longmapsto \mathbf{error}$$

Note that the transitivity between $\langle E_1, \sigma'^{E_2}\rangle \circ \varepsilon_{\sigma^?}$ fails because $\sigma' \neq \sigma$. Thus the results follows immediately.

Case (R?).

$$(\text{R?})\,\frac{\mu;\Xi \vdash t \approx t_{1^*} : ?}{\mu;\Xi \vdash t \approx \varepsilon_{?} t_{1^*} :: ? : ?}$$

Since $t \parallel \mu \longrightarrow$ **error**, we know by the induction hypothesis on $\mu;\Xi \vdash t \approx t_{1^*} : ?$ that $\Xi \triangleright t_{1^*} \longmapsto$ **error**. Thus the result follows immediately.

□

Lemma 9.9. *If* $\mu;\Xi \vdash t \approx t_* : ?$ *and* $t \parallel \mu \longmapsto t' \parallel \mu'$, *then* $\Xi \triangleright t_* \longmapsto^* \Xi' \triangleright t'_*$ *and* $\mu';\Xi' \vdash t' \approx t'_* : ?$, *for some* t'_*.

Proof. The proof is a straightforward induction on $\mu;\Xi \vdash t_1 \approx t_2 : ?$. We only take into account the rules that can be applied.

Case (Rpt).

$$(\text{Rpt})\,\frac{\mu;\Xi \vdash t_1 \approx t_{1^*} : ? \qquad \mu;\Xi \vdash t_2 \approx t_{2^*} : ?}{\mu;\Xi \vdash \langle t_1, t_2\rangle \approx \varepsilon_{?\times ?}\langle t_{1^*}, t_{2^*}\rangle :: ? : ?}$$

If $t \parallel \mu \longrightarrow t' \parallel \mu'$, then by Lemma 9.5, the result follows immediately. Else, if $t \parallel \mu \longmapsto t' \parallel \mu'$, we have the following two cases:

- $t = \langle t_1, t_2\rangle = f[t_1]$, where $f = \langle [], t_2\rangle$.
  Therefore, we have that $t_1 \parallel \mu \longmapsto t'_1 \parallel \mu'$.
  By the induction hypothesis, we get that $\Xi \triangleright t_{1^*} \longmapsto^* \Xi' \triangleright t'_{1^*}$ and $\mu';\Xi' \vdash t'_1 \approx t'_{1^*} : ?$. Thus, we know that
  $$\Xi \triangleright \varepsilon_{?\times ?}\langle t_{1^*}, t_{2^*}\rangle :: ? \longmapsto^* \Xi' \triangleright \varepsilon_{?\times ?}\langle t'_{1^*}, t_{2^*}\rangle :: ?$$
  Therefore, the result follows immediately by Rule (Rpt).
- $t = \langle t_1, t_2\rangle = \langle v_1, t_2\rangle = f[t_2]$, where $f = \langle v_1, []\rangle$. Therefore, we have that $t_2 \parallel \mu \longmapsto t'_2 \parallel \mu'$.
  By the induction hypothesis, we get that $\Xi \triangleright t_{2^*} \longmapsto^* \Xi' \triangleright t'_{2^*}$ and $\mu';\Xi' \vdash t'_2 \approx t'_{2^*} : ?$. Since $\mu;\Xi \vdash v_1 \approx t_{1^*} : ?$, by Lemma 9.4, we know that $\Xi \triangleright t_{1^*} \longmapsto^* \Xi \triangleright v_{1^*}$ and $\mu;\Xi \vdash v_1 \approx v_{1^*} : ?$. Thus, we know that
  $$\Xi \triangleright \varepsilon_{?\times ?}\langle t_{1^*}, t_{2^*}\rangle :: ? \longmapsto^* \Xi \triangleright \varepsilon_{?\times ?}\langle v'_{1^*}, t_{2^*}\rangle :: ? \longmapsto^* \Xi' \triangleright \varepsilon_{?\times ?}\langle v'_{1^*}, t'_{2^*}\rangle :: ?$$
  Therefore, the result follows immediately by Rule (Rpt).

Case (R?).

$$(\text{R?})\,\frac{\mu;\Xi \vdash t_1 \approx t_{1^*} : ?}{\mu;\Xi \vdash t_1 \approx \varepsilon_{?} t_{1^*} :: ? : ?}$$

If $t \parallel \mu \longrightarrow t' \parallel \mu'$, then by Lemma 9.5, the result follows immediately. Else, if $t \parallel \mu \longmapsto t' \parallel \mu'$, we have that $t_1 \parallel \mu \longmapsto t'_1 \parallel \mu'$.

By the induction hypothesis, we get that $\Xi \triangleright t_{1^*} \longmapsto^* \Xi' \triangleright t'_{1^*}$ and $\mu';\Xi' \vdash t'_1 \approx t'_{1^*} : ?$. Thus, we know that

$$\Xi \triangleright \varepsilon_{?} t_{1^*} :: ? \longmapsto^* \Xi' \triangleright \varepsilon_{?} t'_{1^*} :: ?$$

Therefore, the result follows immediately by Rule (R?).

Case (RappL).

$$(\text{RappL})\,\frac{\mu;\Xi \vdash t_1 \approx t_{1^*} : ? \qquad \mu;\Xi \vdash t_2 \approx t_{2^*} : ?}{\mu;\Xi \vdash t_1\ t_2 \approx \mathsf{let}\ x = t_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{?\to ?} x :: ? \to ?)\ y : ?}$$

If $t \parallel \mu \longrightarrow t' \parallel \mu'$, then by Lemma 9.5, the result follows immediately. Else, if $t \parallel \mu \longmapsto t' \parallel \mu'$, we have the following two cases:

- $t = t_1\ t_2 = f[t_1]$, where $f = []\ t_2$. Therefore, we have that $t_1 \parallel \mu \longmapsto t_1 \parallel \mu'$.
  By the induction hypothesis, we get that $\Xi \triangleright t_{1^*} \longmapsto^* \Xi' \triangleright t'_{1^*}$ and $\mu'; \Xi' \vdash t'_1 \approx t'_{1^*} : ?$. Thus, we know that
  $$\Xi \triangleright \mathsf{let}\ x = t_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{? \to ?} x :: ? \to ?)\ y \longmapsto^*$$
  $$\Xi' \triangleright \mathsf{let}\ x = t'_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{? \to ?} x :: ? \to ?)\ y$$
  Therefore, the result follows immediately by Rule (RappL).
- $t = t_1\ t_2 = v_1\ t_2 = f[t_2]$, where $f = v_1\ []$. Therefore, we have that $t_2 \parallel \mu \longmapsto t'_2 \parallel \mu'$. By the induction hypothesis, we get that $\Xi \triangleright t_{2^*} \longmapsto^* \Xi' \triangleright t'_{2^*}$ and $\mu'; \Xi' \vdash t'_2 \approx t'_{2^*} : ?$. Since $\mu; \Xi \vdash v_1 \approx t_{1^*} : ?$, by Lemma 9.4, we know that $\Xi \triangleright t_{1^*} \longmapsto^* \Xi \triangleright v_{1^*}$ and $\mu; \Xi \vdash v_1 \approx v_{1^*} : ?$. Thus, we know that
  $$\Xi \triangleright \mathsf{let}\ x = t_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{? \to ?} x :: ? \to ?)\ y \longmapsto^*$$
  $$\Xi \triangleright \mathsf{let}\ x = v_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{? \to ?} x :: ? \to ?)\ y \longmapsto$$
  $$\Xi \triangleright \varepsilon_?(\mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{? \to ?} v_{1^*} :: ? \to ?)\ y) :: ? \longmapsto^*$$
  $$\Xi' \triangleright \varepsilon_?(\mathsf{let}\ y = t'_{2^*}\ \mathsf{in}\ (\varepsilon_{? \to ?} v_{1^*} :: ? \to ?)\ y) :: ?$$
  Therefore, the result follows immediately by Rules (RappR) and (R?).

Case (RappR).

$$\text{(RappR)}\ \frac{\mu; \Xi \vdash v_1 \approx v_{1^*} : ? \qquad \mu; \Xi \vdash t_2 \approx t_{2^*} : ?}{\mu; \Xi \vdash v_1\ t_2 \approx \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{? \to ?} v_{1^*} :: ? \to ?)\ y : ?}$$

If $t \parallel \mu \longrightarrow t' \parallel \mu'$, then by Lemma 9.5, the result follows immediately. Else, if $t \parallel \mu \longmapsto t' \parallel \mu'$, we know that $t = v_1\ t_2 = f[t_2]$, where $f = v_1\ []$. Therefore, we have that $t_2 \parallel \mu \longmapsto t'_2 \parallel \mu'$. By the induction hypothesis, we get that $\Xi \triangleright t_{2^*} \longmapsto^* \Xi' \triangleright t'_{2^*}$ and $\mu'; \Xi' \vdash t'_2 \approx t'_{2^*} : ?$. Thus, we know that

$$\Xi \triangleright \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{? \to ?} v_{1^*} :: ? \to ?)\ y \longmapsto^*$$
$$\Xi' \triangleright \mathsf{let}\ y = t'_{2^*}\ \mathsf{in}\ (\varepsilon_{? \to ?} v_{1^*} :: ? \to ?)\ y$$

Therefore, the result follows immediately by Rule (RappR).

Case (Rpi).

$$\text{(Rpi)}\ \frac{\mu; \Xi \vdash t_1 \approx t_{1^*} : ?}{\mu; \Xi \vdash \pi_i(t_1) \approx \pi_i(\varepsilon_{? \times ?} t_{1^*} :: ? \times ?) : ?}$$

If $t \parallel \mu \longrightarrow t' \parallel \mu'$, then by Lemma 9.5, the result follows immediately. Else, if $t \parallel \mu \longmapsto t' \parallel \mu'$, we know that $t = \pi_i(t_1) = f[t_1]$, where $\pi_i([])$.

Therefore, we have that $t_1 \parallel \mu \longmapsto t'_1 \parallel \mu'$.

By the induction hypothesis, we get that $\Xi \triangleright t_{1^*} \longmapsto^* \Xi' \triangleright t'_{1^*}$ and $\mu'; \Xi' \vdash t'_1 \approx t'_{1^*} : ?$. Thus, we know that

$$\Xi \triangleright \pi_i(\varepsilon_{? \times ?} t_{1^*} :: ? \times ?) \longmapsto^* \Xi' \triangleright \pi_i(\varepsilon_{? \times ?} t'_{1^*} :: ? \times ?)$$

Therefore, the result follows immediately by Rule (Rpi).

Case (Rsed1L).

$$\text{(Rsed1L)}\ \frac{\mu; \Xi \vdash t_1 \approx t_{1^*} : ? \qquad \mu; \Xi \vdash t_2 \approx t_{2^*} : ?}{\mu; \Xi \vdash \{t_1\}_{t_2} \approx \mathsf{let}\ x = t_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{? \to ?} \pi_1(\varepsilon_{? \times ?} y :: ? \times ?) :: ? \to ?)\ x : ?}$$

If $t \parallel \mu \longrightarrow t' \parallel \mu'$, then by Lemma 9.5, the result follows immediately. Else, if $t \parallel \mu \longmapsto t' \parallel \mu'$, we have the following two cases:

- $t = \{t_1\}_{t_2} = f[t_1]$, where $f = \{[]\}_{t_2}$.
  Therefore, we have that $t_1 \parallel \mu \longmapsto t_1' \parallel \mu'$.
  By the induction hypothesis, we get that $\Xi \triangleright t_{1^*} \longmapsto^* \Xi' \triangleright t_{1^*}'$ and $\mu'; \Xi' \vdash t_1' \approx t_{1^*}' : ?$. Thus, we know that

$$\Xi \triangleright \mathsf{let}\ x = t_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{? \to ?}\pi_1(\varepsilon_{? \times ?}y :: ? \times ?) :: ? \to ?)\ x \longmapsto^*$$

$$\Xi' \triangleright \mathsf{let}\ x = t_{1^*}'\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{? \to ?}\pi_1(\varepsilon_{? \times ?}y :: ? \times ?) :: ? \to ?)\ x$$

  Therefore, the result follows immediately by Rule (Rsed1L).
- $t = \{t_1\}_{t_2} = \{v_1\}_{t_2} = f[t_2]$, where $f = \{v_1\}_{[]}$. Therefore, we have that $t_2 \parallel \mu \longmapsto t_2' \parallel \mu'$. By the induction hypothesis, we get that $\Xi \triangleright t_{2^*} \longmapsto^* \Xi' \triangleright t_{2^*}'$ and $\mu'; \Xi' \vdash t_2' \approx t_{2^*}' : ?$. Since $\mu; \Xi \vdash v_1 \approx t_{1^*} : ?$, by Lemma 9.4, we know that $\Xi \triangleright t_{1^*} \longmapsto^* \Xi \triangleright v_{1^*}$ and $\mu; \Xi \vdash v_1 \approx v_{1^*} : ?$. Thus, we know that

$$\Xi \triangleright \mathsf{let}\ x = t_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{? \to ?}\pi_1(\varepsilon_{? \times ?}y :: ? \times ?) :: ? \to ?)\ x \longmapsto^*$$

$$\Xi \triangleright \mathsf{let}\ x = v_{1^*}'\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{? \to ?}\pi_1(\varepsilon_{? \times ?}y :: ? \times ?) :: ? \to ?)\ x \longmapsto$$

$$\Xi \triangleright \varepsilon_?(\mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{? \to ?}\pi_1(\varepsilon_{? \times ?}y :: ? \times ?) :: ? \to ?)\ v_{1^*}') :: ? \longmapsto^*$$

$$\Xi' \triangleright \varepsilon_?(\mathsf{let}\ y = t_{2^*}'\ \mathsf{in}\ (\varepsilon_{? \to ?}\pi_1(\varepsilon_{? \times ?}y :: ? \times ?) :: ? \to ?)\ v_{1^*}') :: ?$$

  Therefore, the result follows immediately by Rules (Rsed1R) and (R?).

Case (Rsed1R).

$$\text{(Rsed1R)}\frac{\mu; \Xi \vdash v_1 \approx v_{1^*} : ? \qquad \mu; \Xi \vdash t_2 \approx t_{2^*} : ?}{\mu; \Xi \vdash \{v_1\}_{t_2} \approx \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{? \to ?}\pi_1(\varepsilon_{? \times ?}y :: ? \times ?) :: ? \to ?)\ v_{1^*} : ?}$$

If $t \parallel \mu \longrightarrow t' \parallel \mu'$, then by Lemma 9.5, the result follows immediately. Else, if $t \parallel \mu \longmapsto t' \parallel \mu'$, we know that $t = \{t_1\}_{t_2} = \{v_1\}_{t_2} = f[t_2]$, where $f = \{v_1\}_{[]}$. Therefore, we have that $t_2 \parallel \mu \longmapsto t_2' \parallel \mu'$. By the induction hypothesis, we get that $\Xi \triangleright t_{2^*} \longmapsto^* \Xi' \triangleright t_{2^*}'$ and $\mu'; \Xi' \vdash t_2' \approx t_{2^*}' : ?$. Thus, we know that

$$\Xi \triangleright \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{? \to ?}\pi_1(\varepsilon_{? \times ?}y :: ? \times ?) :: ? \to ?)\ v_{1^*}' \longmapsto^*$$

$$\Xi' \triangleright \mathsf{let}\ y = t_{2^*}'\ \mathsf{in}\ (\varepsilon_{? \to ?}\pi_1(\varepsilon_{? \times ?}y :: ? \times ?) :: ? \to ?)\ v_{1^*}'$$

Therefore, the result follows immediately by Rule (Rsed1R).

Case (RunsL).

$$\text{(RunsL)}\frac{\mu; \Xi \vdash t_1 \approx t_{1^*} : ? \qquad \mu; \Xi \vdash t_2 \approx t_{2^*} : ? \qquad \mu; \Xi; z : ? \vdash t_3 \approx t_{3^*} : ?}{\mu; \Xi \vdash \mathsf{let}\ \{z\}_{t_1} = t_2\ \mathsf{in}\ t_3 \approx \mathsf{let}\ x = t_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{? \to ?}\pi_2(\varepsilon_{? \times ?}x :: ? \times ?) :: ? \to ?\ y\ \mathsf{in}\ t_{3^*} : ?}$$

If $t \parallel \mu \longrightarrow t' \parallel \mu'$, then by Lemma 9.5, the result follows immediately. Else, if $t \parallel \mu \longmapsto t' \parallel \mu'$, we have the following two cases:

- $t = \mathsf{let}\ \{z\}_{t_1} = t_2\ \mathsf{in}\ t_3 = f[t_1]$, where $f = \mathsf{let}\ \{z\}_{[]} = t_2\ \mathsf{in}\ t_3$.
  Therefore, we have that $t_1 \parallel \mu \longmapsto t_1' \parallel \mu'$.
  By the induction hypothesis, we get that $\Xi \triangleright t_{1^*} \longmapsto^* \Xi' \triangleright t_{1^*}'$ and $\mu'; \Xi' \vdash t_1' \approx t_{1^*}' : ?$. Thus, we know that

$$\Xi \triangleright \mathsf{let}\ x = t_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{? \to ?}\pi_2(\varepsilon_{? \times ?}x :: ? \times ?) :: ? \to ?\ y\ \mathsf{in}\ t_{3^*} \longmapsto^*$$

$$\Xi' \triangleright \mathsf{let}\ x = t_{1^*}'\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{? \to ?}\pi_2(\varepsilon_{? \times ?}x :: ? \times ?) :: ? \to ?\ y\ \mathsf{in}\ t_{3^*}$$

  Therefore, the result follows immediately by Rule (RunsL).

- $t = \mathsf{let}\ \{z\}_{t_1} = t_2\ \mathsf{in}\ t_3 = \mathsf{let}\ \{z\}_{v_1} = t_2\ \mathsf{in}\ t_3 = f[t_2]$, where $f = \mathsf{let}\ \{z\}_{v_1} = []\ \mathsf{in}\ t_3$. Therefore, we have that $t_2 \parallel \mu \longmapsto t_2' \parallel \mu'$. By the induction hypothesis, we get that $\Xi \triangleright t_{2^*} \longmapsto^* \Xi' \triangleright t_{2^*}'$ and $\mu'; \Xi' \vdash t_2' \approx t_{2^*}' : ?$. Since $\mu; \Xi \vdash v_1 \approx t_{1^*} : ?$, by Lemma 9.4, we know that $\Xi \triangleright t_{1^*} \longmapsto^* \Xi \triangleright v_{1^*}$ and $\mu; \Xi \vdash v_1 \approx v_{1^*} : ?$. Thus, we know that

$$\Xi \triangleright \mathsf{let}\ x = t_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}x :: ?\times?) :: ? \to ?\ y\ \mathsf{in}\ t_{3^*} \longmapsto^*$$
$$\Xi \triangleright \mathsf{let}\ x = v_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}x :: ?\times?) :: ? \to ?\ y\ \mathsf{in}\ t_{3^*} \longmapsto$$
$$\Xi \triangleright \varepsilon_?(\mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}v_{1^*} :: ?\times?) :: ? \to ?\ y\ \mathsf{in}\ t_{3^*}) :: ? \longmapsto^*$$
$$\Xi' \triangleright \varepsilon_?(\mathsf{let}\ y = t_{2^*}'\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}v_{1^*} :: ?\times?) :: ? \to ?\ y\ \mathsf{in}\ t_{3^*}) :: ?$$

  Therefore, the result follows immediately by Rules (RunsR) and (R?).

Case (RunsR).

$$\text{(RunsR)}\ \frac{\mu; \Xi \vdash v_1 \approx v_{1^*} : ? \qquad \mu; \Xi \vdash t_2 \approx t_{2^*} : ? \qquad \mu; \Xi; z : ? \vdash t_3 \approx t_{3^*} : ?}{\mu; \Xi \vdash \mathsf{let}\ \{z\}_{v_1} = t_2\ \mathsf{in}\ t_3 \approx \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}v_{1^*} :: ?\times?) :: ? \to ?\ y\ \mathsf{in}\ t_{3^*} : ?}$$

If $t \parallel \mu \longrightarrow t' \parallel \mu'$, then by Lemma 9.5, the result follows immediately. Else, if $t \parallel \mu \longmapsto t' \parallel \mu'$, then we know that $t = \mathsf{let}\ \{z\}_{t_1} = t_2\ \mathsf{in}\ t_3 = \mathsf{let}\ \{z\}_{v_1} = t_2\ \mathsf{in}\ t_3 = f[t_2]$, where $f = \mathsf{let}\ \{z\}_{v_1} = []\ \mathsf{in}\ t_3$. Therefore, we have that $t_2 \parallel \mu \longmapsto t_2' \parallel \mu'$. By the induction hypothesis, we get that $\Xi \triangleright t_{2^*} \longmapsto^* \Xi' \triangleright t_{2^*}'$ and $\mu'; \Xi' \vdash t_2' \approx t_{2^*}' : ?$. Thus, we know that

$$\Xi \triangleright \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}v_{1^*} :: ?\times?) :: ? \to ?\ y\ \mathsf{in}\ t_{3^*} \longmapsto^*$$
$$\Xi' \triangleright \mathsf{let}\ y = t_{2^*}'\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}v_{1^*} :: ?\times?) :: ? \to ?\ y\ \mathsf{in}\ t_{3^*}$$

Therefore, the result follows immediately by Rule (RunsR).

Case (RsG).

$$\text{(RsG)}\ \frac{\Xi; \Gamma, x : ? \vdash t_1 \approx t_1' : ?}{\mu; \Xi \vdash \nu x.t_1 \approx \mathsf{let}\ x = su_\varepsilon\ \mathsf{in}\ t_1' : ?}$$

Since $t = \nu x.t_1$, we know that $t \parallel \mu \longrightarrow t' \parallel \mu'$. Therefore, by Lemma 9.5, the result follows immediately.

Case (Runs).

$$\text{(Runs)}\ \frac{\mu; \Xi \vdash v_1 \approx v_1' : ? \qquad \mu; \Xi \vdash v_2 \approx v_2' : ? \qquad \mu; \Xi; z : ? \vdash t_3 \approx t_3' : ?}{\mu; \Xi \vdash \mathsf{let}\ \{z\}_{v_1} = v_2\ \mathsf{in}\ t_3 \approx \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}v_1' :: ?\times?) :: ? \to ?\ v_2'\ \mathsf{in}\ t_3' : ?}$$

Since $t = \mathsf{let}\ \{z\}_{v_1} = v_2\ \mathsf{in}\ t_3$, we know that $t \parallel \mu \longrightarrow t' \parallel \mu'$. Therefore, by Lemma 9.5, the result follows immediately.

Case (Rapp).

$$\text{(Rapp)}\ \frac{\mu; \Xi \vdash v_1 \approx v_1' : ? \qquad \mu; \Xi \vdash v_2 \approx v_2' : ?}{\mu; \Xi \vdash v_1\ v_2 \approx (\varepsilon_{?\to?}v_1' :: ? \to ?)\ v_2' : ?}$$

Since $t = v_1\ v_2$, we know that $t \parallel \mu \longrightarrow t' \parallel \mu'$. Therefore, by Lemma 9.5, the result follows immediately.

□

Lemma 9.10. *If* $\mu; \Xi \vdash t_1 \approx t_2 : ?$, $\Xi \subseteq \Xi'$ *and* $\Gamma \subseteq \Gamma'$, *then* $\Xi'; \Gamma' \vdash t_1 \approx t_2 : ?$.

Proof. The proof is a straightforward induction on $\mu; \Xi \vdash t_1 \approx t_2 : ?$. □

Lemma 9.11. *If* $\mu; \Xi \vdash t \approx t_* : ?$ *and* $t \parallel \mu \longmapsto$ **error**, *then* $\Xi \triangleright t \longmapsto^*$ **error**.

Proof. The proof is a straightforward induction on $\mu; \Xi \vdash t_1 \approx t_2 : ?$. We only take into account the rules that can be applied ($t \parallel \mu \longmapsto \textbf{error}$).

Case (Rpt).

$$\text{(Rpt)}\frac{\mu; \Xi \vdash t_1 \approx t_{1^*} : ? \qquad \mu; \Xi \vdash t_2 \approx t_{2^*} : ?}{\mu; \Xi \vdash \langle t_1, t_2 \rangle \approx \varepsilon_{?\times?}\langle t_{1^*}, t_{2^*} \rangle :: ? : ?}$$

If $t \parallel \mu \longrightarrow \textbf{error}$, then by Lemma 9.8, the result follows immediately. Else, if

$$t \parallel \mu \longmapsto \textbf{error}$$

, we have the following two cases:

- $t = \langle t_1, t_2 \rangle = f[t_1]$, where $f = \langle [], t_2 \rangle$. Therefore, we have that $t_1 \parallel \mu \longmapsto \textbf{error}$. By the induction hypothesis, we get that $\Xi \triangleright t_{1^*} \longmapsto^* \textbf{error}$. Therefore, the result follows immediately.
- $t = \langle t_1, t_2 \rangle = \langle v_1, t_2 \rangle = f[t_2]$, where $f = \langle v_1, [] \rangle$.
  Therefore, we have that $t_2 \parallel \mu \longmapsto \textbf{error}$. By the induction hypothesis, we get that $\Xi \triangleright t_{2^*} \longmapsto^* \textbf{error}$. Since $\mu; \Xi \vdash v_1 \approx t_{1^*} : ?$, by Lemma 9.4, we know that $\Xi \triangleright t_{1^*} \longmapsto^* \Xi \triangleright v_{1^*}$ and $\mu; \Xi \vdash v_1 \approx v_{1^*} : ?$. Thus, we know that

  $$\Xi \triangleright \varepsilon_{?\times?}\langle t_{1^*}, t_{2^*} \rangle :: ? \longmapsto^* \Xi \triangleright \varepsilon_{?\times?}\langle v'_{1^*}, t_{2^*} \rangle :: ? \longmapsto^* \textbf{error}$$

  Therefore, the result follows immediately.

Case (R?).

$$\text{(R?)}\frac{\mu; \Xi \vdash t_1 \approx t_{1^*} : ?}{\mu; \Xi \vdash t_1 \approx \varepsilon_{?} t_{1^*} :: ? : ?}$$

If $t \parallel \mu \longrightarrow \textbf{error}$, then by Lemma 9.8, the result follows immediately. Else, if

$$t \parallel \mu \longmapsto \textbf{error}$$

, we have that $t_1 \parallel \mu \longmapsto \textbf{error}$. By the induction hypothesis, we get that $\Xi \triangleright t_{1^*} \longmapsto^* \textbf{error}$. Thus, we know that

$$\Xi \triangleright \varepsilon_{?} t_{1^*} :: ? \longmapsto^* \textbf{error}$$

Therefore, the result follows immediately.

Case (RappL).

$$\text{(RappL)}\frac{\mu; \Xi \vdash t_1 \approx t_{1^*} : ? \qquad \mu; \Xi \vdash t_2 \approx t_{2^*} : ?}{\mu; \Xi \vdash t_1\ t_2 \approx \mathsf{let}\ x = t_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{?\to?} x :: ? \to ?)\ y : ?}$$

If $t \parallel \mu \longrightarrow \textbf{error}$, then by Lemma 9.8, the result follows immediately. Else, if

$$t \parallel \mu \longmapsto \textbf{error}$$

, we have the following two cases:

- $t = t_1\ t_2 = f[t_1]$, where $f = []\ t_2$. Therefore, we have that $t_1 \parallel \mu \longmapsto \textbf{error}$. By the induction hypothesis, we get that $\Xi \triangleright t_{1^*} \longmapsto^* \textbf{error}$. Thus, we know that

  $$\Xi \triangleright \mathsf{let}\ x = t_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{?\to?} x :: ? \to ?)\ y \longmapsto^* \textbf{error}$$

  Therefore, the result follows immediately by Rule (RappL).
- $t = t_1\ t_2 = v_1\ t_2 = f[t_2]$, where $f = v_1\ []$.
  Therefore, we have that $t_2 \parallel \mu \longmapsto \textbf{error}$. By the induction hypothesis, we get that $\Xi \triangleright t_{2^*} \longmapsto^* \textbf{error}$. Since $\mu; \Xi \vdash v_1 \approx t_{1^*} : ?$, by Lemma 9.4, we know that $\Xi \triangleright t_{1^*} \longmapsto^* \Xi \triangleright v_{1^*}$ and $\mu; \Xi \vdash v_1 \approx v_{1^*} : ?$. Thus, we know that

  $$\Xi \triangleright \mathsf{let}\ x = t_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{?\to?} x :: ? \to ?)\ y \longmapsto^*$$

$$\Xi \triangleright \text{let } x = v_{1^*} \text{ in let } y = t_{2^*} \text{ in } (\varepsilon_{?\to?}x :: ? \to ?)\ y \longmapsto$$
$$\Xi \triangleright \varepsilon_?(\text{let } y = t_{2^*} \text{ in } (\varepsilon_{?\to?}v_{1^*} :: ? \to ?)\ y) :: ? \longmapsto^* \mathbf{error}$$

Therefore, the result follows immediately.

Case (RappR).

$$\text{(RappR)}\dfrac{\mu;\Xi \vdash v_1 \approx v_{1^*} : ? \qquad \mu;\Xi \vdash t_2 \approx t_{2^*} : ?}{\mu;\Xi \vdash v_1\ t_2 \approx \text{let } y = t_{2^*} \text{ in } (\varepsilon_{?\to?}v_{1^*} :: ? \to ?)\ y : ?}$$

If $t \parallel \mu \longrightarrow \mathbf{error}$, then by Lemma 9.8, the result follows immediately. Else, if

$$t \parallel \mu \longmapsto \mathbf{error}$$

, we know that $t = v_1\ t_2 = f[t_2]$, where $f = v_1\ []$.

Therefore, we have that $t_2 \parallel \mu \longmapsto \mathbf{error}$. By the induction hypothesis, we get that $\Xi \triangleright t_{2^*} \longmapsto^* \mathbf{error}$. Thus, we know that

$$\Xi \triangleright \text{let } y = t_{2^*} \text{ in } (\varepsilon_{?\to?}v_{1^*} :: ? \to ?)\ y \longmapsto^* \mathbf{error}$$

Therefore, the result follows immediately.

Case (Rpi).

$$\text{(Rpi)}\dfrac{\mu;\Xi \vdash t_1 \approx t_{1^*} : ?}{\mu;\Xi \vdash \pi_i(t_1) \approx \pi_i(\varepsilon_{?\times?}t_{1^*} :: ? \times ?) : ?}$$

If $t \parallel \mu \longrightarrow t' \parallel \mu'$, then by Lemma 9.5, the result follows immediately. Else, if $t \parallel \mu \longmapsto t' \parallel \mu'$, we know that $t = \pi_i(t_1) = f[t_1]$, where $\pi_i([])$.

Therefore, we have that $t_1 \parallel \mu \longmapsto t_1' \parallel \mu'$.

By the induction hypothesis, we get that $\Xi \triangleright t_{1^*} \longmapsto^* \Xi' \triangleright t_{1^*}'$ and $\mu';\Xi' \vdash t_1' \approx t_{1^*}' : ?$. Thus, we know that

$$\Xi \triangleright \pi_i(\varepsilon_{?\times?}t_{1^*} :: ? \times ?) \longmapsto^* \Xi' \triangleright \pi_i(\varepsilon_{?\times?}t_{1^*}' :: ? \times ?)$$

Therefore, the result follows immediately by Rule (Rpi).

Case (Rsed1L).

$$\text{(Rsed1L)}\dfrac{\mu;\Xi \vdash t_1 \approx t_{1^*} : ? \qquad \mu;\Xi \vdash t_2 \approx t_{2^*} : ?}{\mu;\Xi \vdash \{t_1\}_{t_2} \approx \text{let } x = t_{1^*} \text{ in let } y = t_{2^*} \text{ in } (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ? \times ?) :: ? \to ?)\ x : ?}$$

If $t \parallel \mu \longrightarrow \mathbf{error}$, then by Lemma 9.8, the result follows immediately. Else, if

$$t \parallel \mu \longmapsto \mathbf{error}$$

, we have the following two cases:

- $t = \{t_1\}_{t_2} = f[t_1]$, where $f = \{[]\}_{t_2}$. Therefore, we have that $t_1 \parallel \mu \longmapsto \mathbf{error}$. By the induction hypothesis, we get that $\Xi \triangleright t_{1^*} \longmapsto^* \mathbf{error}$. Thus, we know that

  $$\Xi \triangleright \text{let } x = t_{1^*} \text{ in let } y = t_{2^*} \text{ in } (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ? \times ?) :: ? \to ?)\ x \longmapsto^* \mathbf{error}$$

  Therefore, the result follows immediately.
- $t = \{t_1\}_{t_2} = \{v_1\}_{t_2} = f[t_2]$, where $f = \{v_1\}_{[]}$.
  Therefore, we have that $t_2 \parallel \mu \longmapsto \mathbf{error}$. By the induction hypothesis, we get that $\Xi \triangleright t_{2^*} \longmapsto^* \mathbf{error}$. Since $\mu;\Xi \vdash v_1 \approx t_{1^*} : ?$, by Lemma 9.4, we know that $\Xi \triangleright t_{1^*} \longmapsto^* \Xi \triangleright v_{1^*}$ and $\mu;\Xi \vdash v_1 \approx v_{1^*} : ?$. Thus, we know that

  $$\Xi \triangleright \text{let } x = t_{1^*} \text{ in let } y = t_{2^*} \text{ in } (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ? \times ?) :: ? \to ?)\ x \longmapsto^*$$
  $$\Xi \triangleright \text{let } x = v_{1^*}' \text{ in let } y = t_{2^*} \text{ in } (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ? \times ?) :: ? \to ?)\ x \longmapsto$$
  $$\Xi \triangleright \varepsilon_?(\text{let } y = t_{2^*} \text{ in } (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ? \times ?) :: ? \to ?)\ v_{1^*}') :: ? \longmapsto^* \mathbf{error}$$

Therefore, the result follows immediately.

Case (Rsed1R).

$$(\text{Rsed1R})\ \frac{\mu;\Xi \vdash v_1 \approx v_{1^*} : ? \qquad \mu;\Xi \vdash t_2 \approx t_{2^*} : ?}{\mu;\Xi \vdash \{v_1\}_{t_2} \approx \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ? \times ?) :: ? \to ?)\ v_{1^*} : ?}$$

If $t \parallel \mu \longrightarrow \textbf{error}$, then by Lemma 9.8, the result follows immediately. Else, if

$$t \parallel \mu \longmapsto \textbf{error}$$

, we know that $t = \{t_1\}_{t_2} = \{v_1\}_{t_2} = f[t_2]$, where $f = \{v_1\}_{[]}$. Therefore, we have that

$$t_2 \parallel \mu \longmapsto \textbf{error}$$

By the induction hypothesis, we get that $\Xi \triangleright t_{2^*} \longmapsto^* \textbf{error}$. Thus, we know that

$$\Xi \triangleright \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ? \times ?) :: ? \to ?)\ v'_{1^*} \longmapsto^* \textbf{error}$$

Therefore, the result follows immediately.

Case (RunsL).

$$(\text{RunsL})\ \frac{\mu;\Xi \vdash t_1 \approx t_{1^*} : ? \qquad \mu;\Xi \vdash t_2 \approx t_{2^*} : ? \qquad \mu;\Xi;z : ? \vdash t_3 \approx t_{3^*} : ?}{\mu;\Xi \vdash \mathsf{let}\ \{z\}_{t_1} = t_2\ \mathsf{in}\ t_3 \approx \mathsf{let}\ x = t_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}x :: ? \times ?) :: ? \to ?\ y\ \mathsf{in}\ t_{3^*} : ?}$$

If $t \parallel \mu \longrightarrow \textbf{error}$, then by Lemma 9.8, the result follows immediately. Else, if

$$t \parallel \mu \longmapsto \textbf{error}$$

, we have the following two cases:

- $t = \mathsf{let}\ \{z\}_{t_1} = t_2\ \mathsf{in}\ t_3 = f[t_1]$, where $f = \mathsf{let}\ \{z\}_{[]} = t_2\ \mathsf{in}\ t_3$. Therefore, we have that

  $$t_1 \parallel \mu \longmapsto \textbf{error}$$

  By the induction hypothesis, we get that $\Xi \triangleright t_{1^*} \longmapsto^* \textbf{error}$. Thus, we know that

  $$\Xi \triangleright \mathsf{let}\ x = t_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}x :: ? \times ?) :: ? \to ?\ y\ \mathsf{in}\ t_{3^*} \longmapsto^* \textbf{error}$$

  Therefore, the result follows immediately.
- $t = \mathsf{let}\ \{z\}_{t_1} = t_2\ \mathsf{in}\ t_3 = \mathsf{let}\ \{z\}_{v_1} = t_2\ \mathsf{in}\ t_3 = f[t_2]$, where $f = \mathsf{let}\ \{z\}_{v_1} = []\ \mathsf{in}\ t_3$. Therefore, we have that

  $$t_2 \parallel \mu \longmapsto \textbf{error}$$

  By the induction hypothesis, we get that $\Xi \triangleright t_{2^*} \longmapsto^* \textbf{error}$. Since $\mu;\Xi \vdash v_1 \approx t_{1^*} : ?$, by Lemma 9.4, we know that $\Xi \triangleright t_{1^*} \longmapsto^* \Xi \triangleright v_{1^*}$ and $\mu;\Xi \vdash v_1 \approx v_{1^*} : ?$. Thus, we know that

  $$\Xi \triangleright \mathsf{let}\ x = t_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}x :: ? \times ?) :: ? \to ?\ y\ \mathsf{in}\ t_{3^*} \longmapsto^*$$

  $$\Xi \triangleright \mathsf{let}\ x = v_{1^*}\ \mathsf{in}\ \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}x :: ? \times ?) :: ? \to ?\ y\ \mathsf{in}\ t_{3^*} \longmapsto$$

  $$\Xi \triangleright \varepsilon_?(\mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}v_{1^*} :: ? \times ?) :: ? \to ?\ y\ \mathsf{in}\ t_{3^*}) :: ? \longmapsto^* \textbf{error}$$

  Therefore, the result follows immediately.

Case (RunsR).

$$(\text{RunsR})\ \frac{\mu;\Xi \vdash v_1 \approx v_{1^*} : ? \qquad \mu;\Xi \vdash t_2 \approx t_{2^*} : ? \qquad \mu;\Xi;z : ? \vdash t_3 \approx t_{3^*} : ?}{\mu;\Xi \vdash \mathsf{let}\ \{z\}_{v_1} = t_2\ \mathsf{in}\ t_3 \approx \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}v_{1^*} :: ? \times ?) :: ? \to ?\ y\ \mathsf{in}\ t_{3^*} : ?}$$

If $t \parallel \mu \longrightarrow \textbf{error}$, then by Lemma 9.8, the result follows immediately. Else, if

$$t \parallel \mu \longmapsto \textbf{error}$$

, then we know that $t = \mathsf{let}\ \{z\}_{t_1} = t_2\ \mathsf{in}\ t_3 = \mathsf{let}\ \{z\}_{v_1} = t_2\ \mathsf{in}\ t_3 = f[t_2]$, where $f = \mathsf{let}\ \{z\}_{v_1} = []\ \mathsf{in}\ t_3$. Therefore, we have that

$$t_2 \parallel \mu \longmapsto \textbf{error}$$

By the induction hypothesis, we get that $\Xi \triangleright t_{2^*} \longmapsto^* \textbf{error}$ Thus, we know that

$$\Xi \triangleright \mathsf{let}\ y = t_{2^*}\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}v_{1^*} :: ?\times?) :: ?\to?\ y\ \mathsf{in}\ t_{3^*} \longmapsto^* \textbf{error}$$

Therefore, the result follows immediately.

Case (Runs).

$$\text{(Runs)}\frac{\mu;\Xi \vdash v_1 \approx v_1' : ? \qquad \mu;\Xi \vdash v_2 \approx v_2' : ? \qquad \mu;\Xi;z : ? \vdash t_3 \approx t_3' : ?}{\mu;\Xi \vdash \mathsf{let}\ \{z\}_{v_1} = v_2\ \mathsf{in}\ t_3 \approx \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}v_1' :: ?\times?) :: ?\to?\ v_2'\ \mathsf{in}\ t_3' : ?}$$

Since $t = \mathsf{let}\ \{z\}_{v_1} = v_2\ \mathsf{in}\ t_3$, we know that $t \parallel \mu \longrightarrow \textbf{error}$. Therefore, by Lemma 9.8, the result follows immediately.

∎

Lemma 9.12. *If* $\varepsilon = \langle E, E\rangle$, *then* $\varepsilon \circ \varepsilon = \varepsilon$.

Proof. Straightforward induction on the shape of the evidence ε. ∎

Lemma 9.13. *If* $\Xi;\Gamma \vdash \lceil t\rceil_\varepsilon : G$ *then* $G = ?$.

Proof. Straightforward induction on the syntax of t. ∎

Lemma 11.2. *If* $\mu;\Xi;\Gamma \vdash t \approx t_\varepsilon : ?$ *then* $\Xi;\Gamma \vdash t_\varepsilon : ?$.

Proof. Direct by Lemma 9.13. ∎

Lemma 9.14. *If* t *is closed* λ_{seal} *term, then* $\cdot;\cdot;\cdot \vdash \lceil t\rceil_\varepsilon : ?$.

Proof. Straightforward induction on the syntax of t. ∎

Lemma 9.15. $\Xi \triangleright su_\varepsilon \longmapsto^* \Xi, \sigma := ? \triangleright su_\varepsilon^\sigma$, *where* $\sigma := ? \notin \Xi$.

Proof. Following the reduction rules of GSF. ∎

Lemma 9.16 (Substitution preserves). *If* $\mu;\Xi;\Gamma, x : ? \vdash t \approx t^* : ?$ *and* $\mu;\Xi;\Gamma \vdash v \approx v^* : ?$, *then* $\mu;\Xi;\Gamma \vdash t[v/x] \approx t^*[v^*/x] : ?$.

Proof. The proof is a straightforward induction on the derivation of $\mu;\Xi;\Gamma, x : ? \vdash t \approx t^* : ?$.

Case (Rx).

$$\text{(Rx)}\frac{x : ? \in \Gamma, x : ?}{\mu;\Xi;\Gamma, x : ? \vdash x \approx x : ?}$$

We have that $t = x$ and $t^* = x$. By the definition of substitution, we have that $x[v/x] = v$ and $x[v^*/x] = v^*$. Therefore, we are required to prove that $\mu;\Xi;\Gamma \vdash v \approx v^* : ?$, which follows by the premise.

If we have

$$\text{(Rx)}\frac{y : ? \in \Gamma, x : ?}{\mu;\Xi;\Gamma, x : ? \vdash y \approx y : ?}$$

We have that $t = y$ and $t^* = y$. By the definition of substitution, we have that $y[v/x] = y$ and $y[v^*/x] = y$. Therefore, we are required to prove that $\mu;\Xi;\Gamma \vdash y \approx y : ?$, which follows by the premise $\mu;\Xi;\Gamma, x : ? \vdash y \approx y : ?$ and Lemma 9.10.

Case (Rb).

$$\text{(Rb)}\frac{ty(b) = B}{\mu; \Xi; \Gamma, x : ? \vdash b \approx \varepsilon_B b :: ? : ?}$$

We have that $t = b$ and $t^* = \varepsilon_B b :: ?$. By the definition of substitution, we have that $b[v/x] = b$ and $\varepsilon_B b :: ?[v^*/x] = \varepsilon_B b :: ?$. Therefore, we are required to prove that $\mu; \Xi; \Gamma \vdash b \approx \varepsilon_B b :: ? : ?$, which follows by the premise $\mu; \Xi; \Gamma, x : ? \vdash b \approx \varepsilon_B b :: ? : ?$ and Lemma 9.10.

Case (Ru).

$$\text{(Ru)}\frac{\mu; \Xi; \Gamma, x : ? \vdash v_1 \approx \varepsilon_D u :: ? : ?}{\mu; \Xi; \Gamma, x : ? \vdash v_1 \approx \varepsilon_D(\varepsilon_D u :: D) :: ? : ?}$$

We have that $t = v_1$ and $t^* = \varepsilon_D(\varepsilon_D u :: D) :: ?$. By the definition of substitution, we have that $(\varepsilon_D(\varepsilon_D u :: D) :: ?)[v^*/x] = \varepsilon_D(\varepsilon_D u[v^*/x] :: D) :: ?$. Therefore, we are required to prove that $\mu; \Xi; \Gamma \vdash v_1[v/x] \approx \varepsilon_D(\varepsilon_D u[v^*/x] :: D) :: ? : ?$, or what is the same $\mu; \Xi; \Gamma \vdash v_1[v/x] \approx (\varepsilon_D u[v^*/x] :: ?) : ?$ which follows by the induction hypothesis on $\mu; \Xi; \Gamma, x : ? \vdash v_1 \approx \varepsilon_D u :: ? : ?$.

Case (Rs).

$$\text{(Rs)}\frac{\sigma := ? \in \Xi}{\mu; \Xi; \Gamma, x : ? \vdash \sigma \approx su^{\sigma}_{\varepsilon} : ?}$$

We have that $t = \sigma$ and $t^* = su^{\sigma}_{\varepsilon}$. By the definition of substitution, we have that $\sigma[v/x] = \sigma$ and $su^{\sigma}_{\varepsilon}[v^*/x] = su^{\sigma}_{\varepsilon}$. Therefore, we are required to prove that $\mu; \Xi; \Gamma \vdash \sigma \approx su^{\sigma}_{\varepsilon} : ?$, which follows by the premise $\mu; \Xi; \Gamma, x : ? \vdash \sigma \approx su^{\sigma}_{\varepsilon} : ?$ and Lemma 9.10.

Case (Rp).

$$\text{(Rp)}\frac{\mu; \Xi; \Gamma, x : ? \vdash v_1 \approx \varepsilon_{D_1} u_1 :: ? : ? \qquad \mu; \Xi; \Gamma, x : ? \vdash v_2 \approx \varepsilon_{D_2} u_2 :: ? : ?}{\mu; \Xi; \Gamma, x : ? \vdash \langle v_1, v_2 \rangle \approx \varepsilon_{D_1 \times D_2} \langle u_1, u_2 \rangle :: ? : ?}$$

We have that $t = \langle v_1, v_2 \rangle$ and $t^* = \varepsilon_{D_1 \times D_2} \langle u_1, u_2 \rangle :: ?$. By the definition of substitution, we have that $\langle v_1, v_2 \rangle[v/x] = \langle v_1[v/x], v_2[v/x] \rangle$ and $(\varepsilon_{D_1 \times D_2} \langle u_1, u_2 \rangle :: ?)[v^*/x] = \varepsilon_{D_1 \times D_2} \langle u_1[v^*/x], u_2[v^*/x] \rangle :: ?$. Therefore, we are required to prove that $\mu; \Xi; \Gamma \vdash \langle v_1[v/x], v_2[v/x] \rangle \approx \varepsilon_{D_1 \times D_2} \langle u_1[v^*/x], u_2[v^*/x] \rangle :: ? : ?$, or what is the same by Rule (Rp) that $\mu; \Xi; \Gamma \vdash v_1[v/x] \approx \varepsilon_{D_1} u_1[v^*/x] :: ? : ?$ and $\mu; \Xi; \Gamma \vdash v_2[v/x] \approx \varepsilon_{D_2} u_2[v^*/x] :: ? : ?$. By the induction hypothesis on $\mu; \Xi; \Gamma, x : ? \vdash v_1 \approx \varepsilon_{D_1} u_1 :: ? : ?$ and $\mu; \Xi; \Gamma, x : ? \vdash v_2 \approx \varepsilon_{D_2} u_2 :: ? : ?$ the result follows immediately.

Case (Rλ).

$$\text{(R}\lambda\text{)}\frac{\Xi; \Gamma, x : ?, y : ? \vdash t_1 \approx t_2 : ?}{\mu; \Xi; \Gamma, x : ? \vdash (\lambda y.t_1) \approx \varepsilon_{? \to ?}(\lambda y.t_2) :: ? : ?}$$

We have that $t = (\lambda y.t_1)$ and $t^* = \varepsilon_{? \to ?}(\lambda y.t_2) :: ?$. By the definition of substitution, we have that $(\lambda y.t_1)[v/x] = (\lambda y.t_1[v/x])$ and $(\varepsilon_{? \to ?}(\lambda y.t_2) :: ?)[v^*/x] = \varepsilon_{? \to ?}(\lambda y.t_2[v^*/x]) :: ?$. Therefore, we are required to prove that $\mu; \Xi; \Gamma \vdash (\lambda y.t_1[v/x]) \approx \varepsilon_{? \to ?}(\lambda y.t_2[v^*/x]) :: ? : ?$, or what is the same $\mu; \Xi; \Gamma, y : ? \vdash t_1[v/x] \approx t_2[v^*/x] : ?$ which follows by the induction hypothesis on $\mu; \Xi; \Gamma, x : ?, y : ? \vdash t_1 \approx t_2 : ?$.

Case (Rpt).

$$\text{(Rpt)}\frac{\mu; \Xi; \Gamma, x : ? \vdash t_1 \approx t_1' : ? \qquad \mu; \Xi; \Gamma, x : ? \vdash t_2 \approx t_2' : ?}{\mu; \Xi; \Gamma, x : ? \vdash \langle t_1, t_2 \rangle \approx \varepsilon_{? \times ?} \langle t_1', t_2' \rangle :: ? : ?}$$

We have that $t = \langle t_1, t_2 \rangle$ and $t^* = \varepsilon_{? \times ?} \langle t_1', t_2' \rangle :: ?$. By the definition of substitution, we have that $\langle t_1, t_2 \rangle[v/x] = \langle t_1[v/x], t_2[v/x] \rangle$ and $(\varepsilon_{? \times ?} \langle t_1', t_2' \rangle :: ?)[v^*/x] = (\varepsilon_{? \times ?} \langle t_1'[v^*/x], t_2'[v^*/x] \rangle :: ?)$. Therefore, we are required to prove that $\mu; \Xi; \Gamma \vdash \langle t_1[v/x], t_2[v/x] \rangle \approx (\varepsilon_{? \times ?} \langle t_1'[v^*/x], t_2'[v^*/x] \rangle :: ?) : ?$, or what is the same by Rule (Rpt) that $\mu; \Xi; \Gamma \vdash t_1[v/x] \approx t_1'[v^*/x] : ?$ and $\mu; \Xi; \Gamma \vdash t_2[v/x] \approx$

$t_2'[v^*/x] : ?$. By the induction hypothesis on $\mu;\Xi;\Gamma, x : ? \vdash t_1 \approx t_1' : ?$ and $\mu;\Xi;\Gamma, x : ? \vdash t_2 \approx t_2' : ?$ the result follows immediately.

Case (Rapp).

$$\text{(Rapp)}\frac{\mu;\Xi;\Gamma, x : ? \vdash v_1 \approx v_1' : ? \qquad \mu;\Xi;\Gamma, x : ? \vdash v_2 \approx v_2' : ?}{\mu;\Xi;\Gamma, x : ? \vdash v_1\ v_2 \approx (\varepsilon_{?\to?} v_1' :: ? \to ?)\ v_2' : ?}$$

We have that $t = v_1\ v_2$ and $t^* = (\varepsilon_{?\to?} v_1' :: ? \to ?)\ v_2'$. By the definition of substitution, we have that

$$(v_1\ v_2)[v/x] = v_1[v/x]\ v_2[v/x]$$

and

$$((\varepsilon_{?\to?} v_1' :: ? \to ?)\ v_2')[v^*/x] = (\varepsilon_{?\to?} v_1'[v^*/x] :: ? \to ?)\ v_2'[v^*/x]$$

Therefore, we are required to prove that

$$\mu;\Xi;\Gamma \vdash v_1[v/x]\ v_2[v/x] \approx (\varepsilon_{?\to?} v_1'[v^*/x] :: ? \to ?)\ v_2'[v^*/x] : ?$$

, or what is the same by Rule (Rapp) that $\mu;\Xi;\Gamma \vdash v_1[v/x] \approx v_1'[v^*/x] : ?$ and $\mu;\Xi;\Gamma \vdash v_2[v/x] \approx v_2'[v^*/x] : ?$. By the induction hypothesis on $\mu;\Xi;\Gamma, x : ? \vdash v_1 \approx v_1' : ?$ and $\mu;\Xi;\Gamma, x : ? \vdash v_2 \approx v_2' : ?$ the result follows immediately.

Case (RappL).

$$\text{(RappL)}\frac{\mu;\Xi;\Gamma, x : ? \vdash t_1 \approx t_1' : ? \qquad \mu;\Xi;\Gamma, x : ? \vdash t_2 \approx t_2' : ?}{\mu;\Xi;\Gamma, x : ? \vdash t_1\ t_2 \approx \mathsf{let}\ z = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?} z :: ? \to ?)\ y : ?}$$

We have that $t = t_1\ t_2$ and $t^* = \mathsf{let}\ z = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?} z :: ? \to ?)\ y$. By the definition of substitution, we have that

$$(t_1\ t_2)[v/x] = t_1[v/x]\ t_2[v/x]$$

and

$$(\mathsf{let}\ z = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?} z :: ? \to ?)\ y)[v^*/x] = \mathsf{let}\ z = t_1'[v^*/x]\ \mathsf{in}\ \mathsf{let}\ y = t_2'[v^*/x]\ \mathsf{in}\ (\varepsilon_{?\to?} z :: ? \to ?)\ y$$

Therefore, we are required to prove that

$$\mu;\Xi;\Gamma \vdash t_1[v/x]\ t_2[v/x] \approx \mathsf{let}\ z = t_1'[v^*/x]\ \mathsf{in}\ \mathsf{let}\ y = t_2'[v^*/x]\ \mathsf{in}\ (\varepsilon_{?\to?} z :: ? \to ?)\ y : ?$$

, or what is the same by Rule (RappL) that $\mu;\Xi;\Gamma \vdash t_1[v/x] \approx t_1'[v^*/x] : ?$ and $\mu;\Xi;\Gamma \vdash t_2[v/x] \approx t_2'[v^*/x] : ?$. By the induction hypothesis on $\mu;\Xi;\Gamma, x : ? \vdash t_1 \approx t_1' : ?$ and $\mu;\Xi;\Gamma, x : ? \vdash t_2 \approx t_2' : ?$ the result follows immediately.

Case (RappR).

$$\text{(RappR)}\frac{\mu;\Xi;\Gamma, x : ? \vdash v_1 \approx v_1' : ? \qquad \mu;\Xi;\Gamma, x : ? \vdash t_2 \approx t_2' : ?}{\mu;\Xi;\Gamma, x : ? \vdash v_1\ t_2 \approx \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?} v_1' :: ? \to ?)\ y : ?}$$

We have that $t = v_1\ t_2$ and $t^* = \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?} v_1' :: ? \to ?)\ y$. By the definition of substitution, we have that

$$(v_1\ t_2)[v/x] = v_1[v/x]\ t_2[v/x]$$

and

$$(\mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?} v_1' :: ? \to ?)\ y)[v^*/x] = \mathsf{let}\ y = t_2'[v^*/x]\ \mathsf{in}\ (\varepsilon_{?\to?} v_1'[v^*/x] :: ? \to ?)\ y$$

Therefore, we are required to prove that

$$\mu;\Xi;\Gamma \vdash v_1[v/x]\ t_2[v/x] \approx \mathsf{let}\ y = t_2'[v^*/x]\ \mathsf{in}\ (\varepsilon_{?\to?} v_1'[v^*/x] :: ? \to ?)\ y : ?$$

, or what is the same by Rule (RappR) that $\mu;\Xi;\Gamma \vdash v_1[v/x] \approx v_1'[v^*/x] : ?$ and $\mu;\Xi;\Gamma \vdash t_2[v/x] \approx t_2'[v^*/x] : ?$. By the induction hypothesis on $\mu;\Xi;\Gamma, x : ? \vdash v_1 \approx v_1' : ?$ and $\mu;\Xi;\Gamma, x : ? \vdash t_2 \approx t_2' : ?$ the result follows immediately.

Case (R?).

$$\text{(R?)}\,\frac{\mu;\Xi;\Gamma,x:?\vdash t\approx t':?}{\mu;\Xi;\Gamma,x:?\vdash t\approx \varepsilon_? t'::?:?}$$

We have that $t^* = \varepsilon_? t' :: ?$. By the definition of substitution, we have that

$$(\varepsilon_? t' :: ?)[v^*/x] = \varepsilon_? t'[v^*/x] :: ?$$

Therefore, we are required to prove that

$$\mu;\Xi;\Gamma\vdash t[v/x]\approx \varepsilon_? t'[v^*/x] :: ?:?$$

, or what is the same by Rule (R?) that $\mu;\Xi;\Gamma\vdash t[v/x]\approx t'[v^*/x] : ?$. By the induction hypothesis on $\mu;\Xi;\Gamma,x:?\vdash t\approx t':?$ the result follows immediately.

Case (Rpi).

$$\text{(Rpi)}\,\frac{\mu;\Xi;\Gamma,x:?\vdash t''\approx t':?}{\mu;\Xi;\Gamma,x:?\vdash \pi_i(t'')\approx \pi_i(\varepsilon_{?\times?}t'::?\times?):?}$$

We have that $t=\pi_i(t'')$ and $t^* = \pi_i(\varepsilon_{?\times?}t'::?\times?)$. By the definition of substitution, we have that

$$\pi_i(t'')[v/x] = \pi_i(t''[v/x])$$

and

$$(\pi_i(\varepsilon_{?\times?}t'::?\times?))[v^*/x] = \pi_i(\varepsilon_{?\times?}t'[v^*/x]::?\times?)$$

Therefore, we are required to prove that

$$\mu;\Xi;\Gamma\vdash \pi_i(t''[v/x])\approx \pi_i(\varepsilon_{?\times?}t'[v^*/x]::?\times?):?$$

, or what is the same by Rule (Rpi) that $\mu;\Xi;\Gamma\vdash t''[v/x]\approx t'[v^*/x]:?$. By the induction hypothesis on $\mu;\Xi;\Gamma,x:?\vdash t''\approx t':?$ the result follows immediately.

Case (RsG).

$$\text{(RsG)}\,\frac{\mu;\Xi;\Gamma,x:?,z:?\vdash t''\approx t':?}{\mu;\Xi;\Gamma,x:?\vdash \nu z.t''\approx \mathsf{let}\ z=su_\varepsilon\ \mathsf{in}\ t':?}$$

We have that $t=\nu z.t''$ and $t^*=\mathsf{let}\ z=su_\varepsilon\ \mathsf{in}\ t'$. By the definition of substitution, we have that

$$(\nu z.t'')[v/x] = \nu z.t''[v/x]$$

and

$$(\mathsf{let}\ z=su_\varepsilon\ \mathsf{in}\ t')[v^*/x] = \mathsf{let}\ z=su_\varepsilon\ \mathsf{in}\ t'[v^*/x]$$

Therefore, we are required to prove that

$$\mu;\Xi;\Gamma\vdash \nu z.t''[v/x]\approx \mathsf{let}\ z=su_\varepsilon\ \mathsf{in}\ t'[v^*/x]:?$$

, or what is the same by Rule (RsG) that $\mu;\Xi;\Gamma,z:?\vdash t''[v/x]\approx t'[v^*/x]:?$. By the induction hypothesis on $\mu;\Xi;\Gamma,x:?,z:?\vdash t''\approx t':?$ the result follows immediately.

Case (Rsed1).

$$\text{(Rsed1)}\,\frac{\mu;\Xi;\Gamma,x:?\vdash v_1\approx v_1':? \qquad \mu;\Xi;\Gamma,x:?\vdash v_2\approx v_2':?}{\mu;\Xi;\Gamma,x:?\vdash \{v_1\}_{v_2}\approx \varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}v_2'::?\times?)::?\to?\ v_1':?}$$

We have that $t=\{v_1\}_{v_2}$ and $t^*=\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}v_2'::?\times?)::?\to?\ v_1'$. By the definition of substitution, we have that

$$\{v_1\}_{v_2}[v/x] = \{v_1[v/x]\}_{v_2[v/x]}$$

and

$$(\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}v_2'::?\times?)::?\to?\ v_1')[v^*/x] = \varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}v_2'[v^*/x]::?\times?)::?\to?\ v_1'[v^*/x]$$

Therefore, we are required to prove that

$$\mu;\Xi;\Gamma \vdash \{v_1[v/x]\}_{v_2[v/x]} \approx \varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}v_2'[v^*/x] :: ?\times?) :: ?\to?\ v_1'[v^*/x] : ?$$

, or what is the same by Rule (Rsed1) that $\mu;\Xi;\Gamma \vdash v_1[v/x] \approx v_1'[v^*/x] : ?$ and $\mu;\Xi;\Gamma \vdash v_2[v/x] \approx v_2'[v^*/x] : ?$. By the induction hypothesis on $\mu;\Xi;\Gamma, x : ? \vdash v_1 \approx v_1' : ?$ and $\mu;\Xi;\Gamma, x : ? \vdash v_2 \approx v_2' : ?$ the result follows immediately.

Case (Rsed1L).

$$\text{(Rsed1L)}\frac{\mu;\Xi;\Gamma, x : ? \vdash t_1 \approx t_1' : ? \qquad \mu;\Xi;\Gamma, x : ? \vdash t_2 \approx t_2' : ?}{\mu;\Xi;\Gamma, x : ? \vdash \{t_1\}_{t_2} \approx \mathsf{let}\ z = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ?\times?) :: ?\to?)\ z : ?}$$

We have that $t = \{t_1\}_{t_2}$ and $t^* = \mathsf{let}\ z = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ?\times?) :: ?\to?)\ z$. By the definition of substitution, we have that

$$\{t_1\}_{t_2}[v/x] = \{t_1[v/x]\}_{t_2[v/x]}$$

and

$$\begin{gathered}(\mathsf{let}\ z = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ?\times?) :: ?\to?)\ z)[v^*/x] = \\ \mathsf{let}\ z = t_1'[v^*/x]\ \mathsf{in}\ \mathsf{let}\ y = t_2'[v^*/x]\ \mathsf{in}\ (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ?\times?) :: ?\to?)\ z\end{gathered}$$

Therefore, we are required to prove that

$$\mu;\Xi;\Gamma \vdash \{t_1[v/x]\}_{t_2[v/x]} \approx \mathsf{let}\ z = t_1'[v^*/x]\ \mathsf{in}\ \mathsf{let}\ y = t_2'[v^*/x]\ \mathsf{in}\ (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ?\times?) :: ?\to?)\ z : ?$$

, or what is the same by Rule (Rsed1L) that $\mu;\Xi;\Gamma \vdash t_1[v/x] \approx t_1'[v^*/x] : ?$ and $\mu;\Xi;\Gamma \vdash t_2[v/x] \approx t_2'[v^*/x] : ?$. By the induction hypothesis on $\mu;\Xi;\Gamma, x : ? \vdash t_1 \approx t_1' : ?$ and $\mu;\Xi;\Gamma, x : ? \vdash t_2 \approx t_2' : ?$ the result follows immediately.

Case (Rsed1R).

$$\text{(Rsed1R)}\frac{\mu;\Xi;\Gamma, x : ? \vdash v_1 \approx v_1' : ? \qquad \mu;\Xi;\Gamma, x : ? \vdash t_2 \approx t_2' : ?}{\mu;\Xi;\Gamma, x : ? \vdash \{v_1\}_{t_2} \approx \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ?\times?) :: ?\to?)\ v_1' : ?}$$

We have that $t = \{v_1\}_{t_2}$ and $t^* = \mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ?\times?) :: ?\to?)\ v_1'$. By the definition of substitution, we have that

$$\{v_1\}_{t_2}[v/x] = \{v_1[v/x]\}_{t_2[v/x]}$$

and

$$\begin{gathered}(\mathsf{let}\ y = t_2'\ \mathsf{in}\ (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ?\times?) :: ?\to?)\ v_1')[v^*/x] = \\ \mathsf{let}\ y = t_2'[v^*/x]\ \mathsf{in}\ (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ?\times?) :: ?\to?)\ v_1'[v^*/x]\end{gathered}$$

Therefore, we are required to prove that

$$\mu;\Xi;\Gamma \vdash \{v_1[v/x]\}_{t_2[v/x]} \approx \mathsf{let}\ y = t_2'[v^*/x]\ \mathsf{in}\ (\varepsilon_{?\to?}\pi_1(\varepsilon_{?\times?}y :: ?\times?) :: ?\to?)\ v_1'[v^*/x] : ?$$

, or what is the same by Rule (Rsed1R) that $\mu;\Xi;\Gamma \vdash v_1[v/x] \approx v_1'[v^*/x] : ?$ and $\mu;\Xi;\Gamma \vdash t_2[v/x] \approx t_2'[v^*/x] : ?$. By the induction hypothesis on $\mu;\Xi;\Gamma, x : ? \vdash v_1 \approx v_1' : ?$ and $\mu;\Xi;\Gamma, x : ? \vdash t_2 \approx t_2' : ?$ the result follows immediately.

Case (Rsed2).

$$\text{(Rsed2)}\frac{\mu;\Xi;\Gamma, x : ? \vdash v' \approx \langle E_1, E_2\rangle u :: ? : ? \qquad \sigma := ? \in \Xi}{\mu;\Xi;\Gamma, x : ? \vdash \{v'\}_\sigma \approx \langle E_1, \sigma^{E_2}\rangle u :: ? : ?}$$

We have that $t = \{v'\}_\sigma$ and $t^* = \langle E_1, \sigma^{E_2}\rangle u :: ?$. By the definition of substitution, we have that

$$\{v'\}_\sigma[v/x] = \{v'[v/x]\}_\sigma$$

and

$$(\langle E_1, \sigma^{E_2}\rangle u :: ?)[v^*/x] = (\langle E_1, \sigma^{E_2}\rangle u[v^*/x] :: ?)$$

Therefore, we are required to prove that

$$\mu;\Xi;\Gamma \vdash \{v'[v/x]\}_\sigma \approx ((\langle E_1, \sigma^{E_2}\rangle u[v^*/x] :: ?) : ?$$

, or what is the same by Rule (Rsed2) that $\mu;\Xi;\Gamma \vdash v'[v/x] \approx \langle E_1, E_2\rangle u[v^*/x] :: ? : ?$. By the induction hypothesis on $\mu;\Xi;\Gamma, x : ? \vdash v' \approx \langle E_1, E_2\rangle u :: ? : ?$ the result follows immediately.

Case (Runs).

$$\text{(Runs)}\frac{\mu;\Xi;\Gamma, x : ? \vdash v_1 \approx v_1' : ? \qquad \mu;\Xi;\Gamma, x : ? \vdash v_2 \approx v_2' : ? \qquad \mu;\Xi;\Gamma, x : ?, z : ? \vdash t_3 \approx t_3' : ?}{\mu;\Xi;\Gamma, x : ? \vdash \mathsf{let}\ \{z\}_{v_1} = v_2\ \mathsf{in}\ t_3 \approx \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}v_1' :: ?\times?) :: ? \to ?\ v_2'\ \mathsf{in}\ t_3' : ?}$$

We have that $t = \mathsf{let}\ \{z\}_{v_1} = v_2\ \mathsf{in}\ t_3$ and $t^* = \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}v_1' :: ?\times?) :: ? \to ?\ v_2'\ \mathsf{in}\ t_3'$. By the definition of substitution, we have that

$$(\mathsf{let}\ \{z\}_{v_1} = v_2\ \mathsf{in}\ t_3)[v/x] = \mathsf{let}\ \{z\}_{v_1[v/x]} = v_2[v/x]\ \mathsf{in}\ t_3[v/x]$$

and

$$(\mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}v_1' :: ?\times?) :: ? \to ?\ v_2'\ \mathsf{in}\ t_3')[v^*/x] =$$

$$\mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}v_1'[v^*/x] :: ?\times?) :: ? \to ?\ v_2'[v^*/x]\ \mathsf{in}\ t_3'[v^*/x]$$

Therefore, we are required to prove that

$$\mu;\Xi;\Gamma \vdash \mathsf{let}\ \{z\}_{v_1[v/x]} = v_2[v/x]\ \mathsf{in}\ t_3[v/x] \approx \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}v_1'[v^*/x] :: ?\times?) :: ? \to ?\ v_2'[v^*/x]\ \mathsf{in}\ t_3'[v^*/x] : ?$$

Or what is the same by Rule (Runs) that $\mu;\Xi;\Gamma \vdash v_1[v/x] \approx v_1'[v^*/x] : ?$, $\mu;\Xi;\Gamma \vdash v_2[v/x] \approx v_2'[v^*/x] : ?$ and $\mu;\Xi;\Gamma, z : ? \vdash t_3[v^*/x] \approx t_3'[v^*/x] : ?$. By the induction hypothesis on $\mu;\Xi;\Gamma, x : ? \vdash v_1 \approx v_1' : ?$, $\mu;\Xi;\Gamma, x : ? \vdash v_2 \approx v_2' : ?$ and $\mu;\Xi;\Gamma, x : ?, z : ? \vdash t_3 \approx t_3' : ?$ the result follows immediately.

Case (RunsL).

$$\text{(RunsL)}\frac{\mu;\Xi;\Gamma, x : ? \vdash t_1 \approx t_1' : ? \qquad \mu;\Xi;\Gamma, x : ? \vdash t_2 \approx t_2' : ? \qquad \mu;\Xi;\Gamma, x : ?, z : ? \vdash t_3 \approx t_3' : ?}{\mu;\Xi;\Gamma, x : ? \vdash \mathsf{let}\ \{z\}_{t_1} = t_2\ \mathsf{in}\ t_3 \approx \mathsf{let}\ w = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}w :: ?\times?) :: ? \to ?\ y\ \mathsf{in}\ t_3' : ?}$$

We have that $t = \mathsf{let}\ \{z\}_{t_1} = t_2\ \mathsf{in}\ t_3$ and

$$t^* = \mathsf{let}\ w = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}w :: ?\times?) :: ? \to ?\ y\ \mathsf{in}\ t_3'$$

By the definition of substitution, we have that

$$(\mathsf{let}\ \{z\}_{t_1} = t_2\ \mathsf{in}\ t_3)[v/x] = \mathsf{let}\ \{z\}_{t_1[v/x]} = t_2[v/x]\ \mathsf{in}\ t_3[v/x]$$

and

$$(\mathsf{let}\ w = t_1'\ \mathsf{in}\ \mathsf{let}\ y = t_2'\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}w :: ?\times?) :: ? \to ?\ y\ \mathsf{in}\ t_3')[v^*/x] =$$

$$\mathsf{let}\ w = t_1'[v^*/x]\ \mathsf{in}\ \mathsf{let}\ y = t_2'[v^*/x]\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}w :: ?\times?) :: ? \to ?\ y\ \mathsf{in}\ t_3'[v^*/x]$$

Therefore, we are required to prove that

$$\Xi;\Gamma \vdash \mathsf{let}\ \{z\}_{t_1[v/x]} = t_2[v/x]\ \mathsf{in}\ t_3[v/x] \approx$$

$$\mathsf{let}\ w = t_1'[v^*/x]\ \mathsf{in}\ \mathsf{let}\ y = t_2'[v^*/x]\ \mathsf{in}\ \mathsf{let}\ z = \varepsilon_{?\to?}\pi_2(\varepsilon_{?\times?}w :: ?\times?) :: ? \to ?\ y\ \mathsf{in}\ t_3'[v^*/x] : ?$$

Or what is the same by Rule (RunsL) that $\mu;\Xi;\Gamma \vdash t_1[v/x] \approx t_1'[v^*/x] : ?$, $\mu;\Xi;\Gamma \vdash t_2[v/x] \approx t_2'[v^*/x] : ?$ and $\mu;\Xi;\Gamma, z : ? \vdash t_3[v^*/x] \approx t_3'[v^*/x] : ?$. By the induction hypothesis on $\mu;\Xi;\Gamma, x : ? \vdash t_1 \approx t_1' : ?$, $\mu;\Xi;\Gamma, x : ? \vdash t_2 \approx t_2' : ?$ and $\mu;\Xi;\Gamma, x : ?, z : ? \vdash t_3 \approx t_3' : ?$ the result follows immediately.

Case (RunsR).

$$\text{(RunsR)}\frac{\mu;\Xi;\Gamma,x:?\vdash v_1\approx v_1':?\qquad \mu;\Xi;\Gamma,x:?\vdash t_2\approx t_2':?\qquad \mu;\Xi;\Gamma,x:?,z:?\vdash t_3\approx t_3':?}{\mu;\Xi;\Gamma,x:?\vdash \mathsf{let}\ \{z\}_{v_1}=t_2\ \mathsf{in}\ t_3\approx \mathsf{let}\ y=t_2'\ \mathsf{in}\ \mathsf{let}\ z=\varepsilon_{?\rightarrow ?}\pi_2(\varepsilon_{?\times ?}v_1'::?\times ?)::?\rightarrow ?\ y\ \mathsf{in}\ t_3':?}$$

We have that $t=\mathsf{let}\ \{z\}_{v_1}=t_2\ \mathsf{in}\ t_3$ and

$$t^*=\mathsf{let}\ y=t_2'\ \mathsf{in}\ \mathsf{let}\ z=\varepsilon_{?\rightarrow ?}\pi_2(\varepsilon_{?\times ?}v_1'::?\times ?)::?\rightarrow ?\ y\ \mathsf{in}\ t_3'$$

By the definition of substitution, we have that

$$(\mathsf{let}\ \{z\}_{v_1}=t_2\ \mathsf{in}\ t_3)[v/x]=\mathsf{let}\ \{z\}_{v_1[v/x]}=t_2[v/x]\ \mathsf{in}\ t_3[v/x]$$

and

$$(\mathsf{let}\ y=t_2'\ \mathsf{in}\ \mathsf{let}\ z=\varepsilon_{?\rightarrow ?}\pi_2(\varepsilon_{?\times ?}v_1'::?\times ?)::?\rightarrow ?\ y\ \mathsf{in}\ t_3')[v^*/x]=$$
$$\mathsf{let}\ y=t_2'[v^*/x]\ \mathsf{in}\ \mathsf{let}\ z=\varepsilon_{?\rightarrow ?}\pi_2(\varepsilon_{?\times ?}v_1'[v^*/x]::?\times ?)::?\rightarrow ?\ y\ \mathsf{in}\ t_3'[v^*/x]$$

Therefore, we are required to prove that

$$\Xi;\Gamma\vdash \mathsf{let}\ \{z\}_{v_1[v/x]}=t_2[v/x]\ \mathsf{in}\ t_3[v/x]\approx$$
$$\mathsf{let}\ y=t_2'[v^*/x]\ \mathsf{in}\ \mathsf{let}\ z=\varepsilon_{?\rightarrow ?}\pi_2(\varepsilon_{?\times ?}v_1'[v^*/x]::?\times ?)::?\rightarrow ?\ y\ \mathsf{in}\ t_3'[v^*/x]:?$$

Or what is the same by Rule (RunsR) that $\mu;\Xi;\Gamma\vdash v_1[v/x]\approx v_1'[v^*/x]:?$, $\mu;\Xi;\Gamma\vdash t_2[v/x]\approx t_2'[v^*/x]:?$ and $\mu;\Xi;\Gamma,z:?\vdash t_3[v^*/x]\approx t_3'[v^*/x]:?$. By the induction hypothesis on $\mu;\Xi;\Gamma,x:?\vdash v_1\approx v_1':?$, $\mu;\Xi;\Gamma,x:?\vdash t_2\approx t_2':?$ and $\mu;\Xi;\Gamma,x:?,z:?\vdash t_3\approx t_3':?$ the result follows immediately.

□

Lemma 11.4. *If* $\mu;\Xi;\Gamma,x:?\vdash t\approx t_\varepsilon:?$ *and* $\mu;\Xi;\Gamma\vdash v\approx v_\varepsilon:?$, *then* $\mu;\Xi;\Gamma\vdash t[v/x]\approx t_\varepsilon[v_\varepsilon/x]:?$.

Proof. Direct by Lemma 9.16. □

The remaining theorems and lemmas are in the main text.

10 GRADUAL EXISTENTIAL TYPES IN GSF

This session presents a motivational example for the extension of GSF with existential directly instead of using the encoding of existential into universal types. Also, we show the translation from GSF$^{\exists}$ to GSF$_{\varepsilon}^{\exists}$ and the proof of the fundamental property for existential types.

10.1 Existential types: primitive or encoded?

The benefit of a direct treatment of existential types can already be appreciated in the fully-static setting, with the simple examples of packages s_1 and s_2 above. Suppose we want to show that s_1 and s_2 are contextually equivalent, *i.e.* indistinguishable by any context. To show this equivalence, it is sufficient to show that the packages are logically related. The proof of this based on the direct interpretation of the existential types is considerably easier and more intuitive than proving that their *encodings* are related. To illustrate this point, we sketch these two proof techniques below in System F.

Proof with primitive existentials. Two packages are logically related at an existential type, if there exists a relation R between values of their representation types, such that their term components respect the relation R. Here, for v_1 and v_2 to respect R means that the following three conditions hold:

- The created semaphores with the operation *bit* are related. In this case, this imposes that $(\mathsf{true}, 1) \in R$.
- If two semaphores are related, then changing their states with the operation *flip* yields related semaphores. Here, applying the *flip* operation of each package s_1 and s_2 to the values true and 1, respectively, yields false and 0. Therefore, $(\mathsf{false}, 0) \in R$. Applying the *flip* operations on these values yields again true and 1, which are related.
- If two semaphores are related, then the Bool value obtained by applying the operation *read* must be the same. It is easy to see that this condition is also satisfied.

Formally, two packages are logically related at an existential type in standard System F (following [Ahmed 2006]):

$$\begin{aligned}\mathcal{V}_\rho[\![\exists X.T]\!] = \quad & \{(\mathsf{pack}\langle T_1, v_1\rangle \text{ as } \exists X.\rho(T), \mathsf{pack}\langle T_2, v_2\rangle \text{ as } \exists X.\rho(T)) \in \mathrm{Atom}_\rho^{=}[\exists X.T] \mid \\ & \exists R \in \textsc{Rel}[T_1, T_2].(v_1, v_2) \in \mathcal{V}_{\rho[X \mapsto (R, T_1, T_2)]}[\![T]\!]\}\end{aligned}$$

By this definition, in order to prove that s_1 is logically related to s_2 at type *Sem*, it is required to show that there exists a relation R between the types Bool and Int such that

$$(v_1, v_2) \in \mathcal{V}_{[X \mapsto (R, \mathsf{Bool}, \mathsf{Int})]}[\![X \times (X \to X) \times (X \to \mathsf{Bool})]\!]$$

Taking $R = \{\langle \mathsf{true}, 1\rangle, \langle \mathsf{false}, 0\rangle\}$, it is easy to check that the implementations of s_1 and s_2 preserve this relation.

Proof with encoded existentials. Using the encoding of *Sem* in terms of universal types in order to prove that s_1 and s_2 are logically related is considerably more complex. First, we have to transform the packages s_1 and s_2 to type abstractions and prove that

$$((\Lambda Y.\lambda f : Sem_{client}.f\ [\mathsf{Bool}]\ v_1), (\Lambda Y.\lambda f : Sem_{client}.f\ [\mathsf{Int}]\ v_2)) \in \mathcal{V}_\rho[\![\forall Y.Sem_{client} \to Y]\!]$$

where $Sem_{client} = \forall X.X \times (X \to X) \times (X \to \mathsf{Bool}) \to Y$. The proof of the above leads us to show that for any type T_1' and T_2', and any relation R' between these types, the following type applications are related:

$$((\Lambda Y.\lambda f : Sem_{client}.f\ [\mathsf{Bool}]\ v_1)\ [T_1'], (\Lambda Y.\lambda f : Sem_{client}.f\ [\mathsf{Int}]\ v_2)\ [T_2']) \in \mathcal{T}_{[Y \mapsto (R', T_1', T_2')]}[\![Sem_{client} \to Y]\!]$$

Several steps further in the proof, we have to show that $(f_1\ [\mathsf{Bool}]\ v_1, f_2\ [\mathsf{Int}]\ v_2) \in \mathcal{T}_{[Y\mapsto(R',T_1',T_2')]}[\![Y]\!]$, for any f_1 and f_2 such that

$$(f_1, f_2) \in \mathcal{V}_{[Y\mapsto(R',T_1',T_2')]}[\![Sem_{client}]\!]$$

Since f_1 and f_2 are related under a universal type, we can instantiate them at any types T_1 and T_2, and any relation Q between these types, keeping the resulting terms related:

$$(f_1\ [T_1], f_2\ [T_2]) \in \mathcal{T}_{[Y\mapsto(R',T_1',T_2'),X\mapsto(Q,T_1,T_2)]}[\![(X \times (X \to X) \times (X \to \mathsf{Bool}) \to Y)]\!]$$

At this point, we can pick the same relation as above, $R = \{\langle \mathsf{true}, 1\rangle, \langle \mathsf{false}, 0\rangle\}$, such that v_1 and v_2 are related.

$$(v_1, v_2) \in \mathcal{V}_{[X\mapsto(R,\mathsf{Bool},\mathsf{Int})]}[\![X \times (X \to X) \times (X \to \mathsf{Bool})]\!]$$

Hence, we can instantiate T_1 and T_2 with the types Bool and Int, and Q with the relation R, obtaining that

$$(f_1\ [\mathsf{Bool}], f_2\ [\mathsf{Int}]) \in \mathcal{T}_{[Y\mapsto(R',T_1',T_2'),X\mapsto(R,\mathsf{Bool},\mathsf{Int})]}[\![(X \times (X \to X) \times (X \to \mathsf{Bool}) \to Y)]\!]$$

In a few more steps, we can instantiate the above with v_1 and v_2, since they are related, finally obtaining the desired result.

As we can see, as part of the second approach (using the encoding) is needed to prove the same that is required by the first approach (directly on existential types) and more; being the second extremely more complex. The equivalence example that we use to illustrate the previous is very simple. But, for instance, Ahmed et al. [2009a] prove challenging cases of equivalences in the presence of abstract data types and mutable references, where the use of the encoding would have hindered the work.

10.2 Translation from $\mathrm{GSF}^\exists$ to $\mathrm{GSF}^\exists_\varepsilon$

Figure 26 shows the translation from $\mathrm{GSF}^\exists$ to $\mathrm{GSF}^\exists_\varepsilon$.

$$\dots$$

$$\text{(Gpacku)}\ \frac{\Delta;\Gamma \vdash v :: G[G'/X] \rightsquigarrow v' : G[G'/X] \qquad \Delta \vdash G'}{\Delta;\Gamma \vdash \mathsf{pack}\langle G', v\rangle \text{ as } \exists X.G \rightsquigarrow \mathsf{packu}\langle G', v'\rangle \text{ as } \exists X.G : \exists X.G}$$

$$\text{(Gpack)}\ \frac{t \neq v \qquad \Delta;\Gamma \vdash t \rightsquigarrow t' : G_1 \qquad \varepsilon = \mathcal{I}(G_1, G[G'/X]) \qquad \Delta \vdash G'}{\Delta;\Gamma \vdash \mathsf{pack}\langle G', t\rangle \text{ as } \exists X.G \rightsquigarrow \mathsf{pack}\langle G', \varepsilon t :: G[G'/X]\rangle \text{ as } \exists X.G : \exists X.G}$$

$$\text{(Gunpack)}\ \frac{\begin{array}{c}\Delta;\Gamma \vdash t_1 \rightsquigarrow t_1' : G_1 \qquad G_1 \twoheadrightarrow \exists X.G_1' \qquad \varepsilon = \mathcal{I}(G_1, \exists X.G_1') \\ \Delta, X;\Gamma, x : G_1' \vdash t_2 \rightsquigarrow t_2' : G_2 \qquad \Delta \vdash G_2\end{array}}{\Delta;\Gamma \vdash \mathsf{unpack}\langle X, x\rangle = t_1 \text{ in } t_2 \rightsquigarrow \mathsf{unpack}\langle X, x\rangle = \varepsilon t_1' :: \exists X.G_1' \text{ in } t_2' : G_2}$$

Fig. 26. Translation from $\mathrm{GSF}^\exists$ to $\mathrm{GSF}^\exists_\varepsilon$

10.3 Properties of $\mathrm{GSF}^\exists$

Proposition 12.1 ($\mathrm{GSF}^\exists$: Precision, inductively). *The inductive definition of type precision given in Figure 17 is equivalent to Definition 6.1.*

Proof. Direct by induction on the type structure of G_1 and G_2. Similar to Prop. 6.2. ◻

Proposition 12.2 ($\mathrm{GSF}^\exists$: Consistency, inductively). *The inductive definition of type consistency given in Figure 17 is equivalent to Definition 6.5.*

Proof. Similar to Prop. 6.6. □

Proposition 12.3 (GSF$^\exists$: Static equivalence for static terms). *Let t be a static term and G a static type ($G = T$). We have $\vdash_S t : T$ if and only if $\vdash t : T$.*

Proof. Smilar to Prop. 6.9. □

Proposition 12.4 (GSF$^\exists$: Static gradual guarantee). *Let t and t' be closed GSF$^\exists$ terms such that $t \sqsubseteq t'$ and $\vdash t : G$. Then $\vdash t' : G'$ and $G \sqsubseteq G'$.*

Proof. Similar to Prop. 6.10. □

10.4 GSF$^\exists$: Parametricity

Theorem 10.1 (Fundamental Property). *If $\Xi; \Delta; \Gamma \vdash t : G$ then $\Xi; \Delta; \Gamma \vdash t \preceq t : G$.*

We follow by induction on the structure of t.

Proof.

Case (**packu**). Then $t = \varepsilon(\mathsf{packu}\langle G', v\rangle \text{ as } \exists X.G'') :: G$, and therefore by the typing rules Epacku and Easc we have that

$$\text{(Epack \& Easc)}\frac{\Xi; \Delta; \Gamma \vdash v : G''[G'/X] \qquad \Xi; \Delta \vdash G' \qquad \varepsilon \Vdash \Xi; \Delta \vdash \exists X.G'' \sim G}{\Xi; \Delta; \Gamma \vdash \varepsilon(\mathsf{packu}\langle G', v\rangle \text{ as } \exists X.G'') :: G : G}$$

Then we have to prove that:

$$\Xi; \Delta; \Gamma \vdash \varepsilon(\mathsf{packu}\langle G', v\rangle \text{ as } \exists X.G'') :: G \preceq \varepsilon(\mathsf{packu}\langle G', v\rangle \text{ as } \exists X.G'') :: G : G$$

By induction hypotheses we already know that $\Xi; \Delta; \Gamma \vdash v \preceq v : G''[G'/X]$. But the result follows directly by Prop 10.2 (Compatibility of packu).

Case (**pack**). Then $t = \mathsf{pack}\langle G', t'\rangle \text{ as } \exists X.G''$, and therefore by the typing rules Epack we have that

$$\text{(Epack)}\frac{\Xi; \Delta; \Gamma \vdash t' : G''[G'/X] \qquad \Xi; \Delta \vdash G'}{\Xi; \Delta; \Gamma \vdash \mathsf{pack}\langle G', t'\rangle \text{ as } \exists X.G'' : \exists X.G''}$$

Then we have to prove that:

$$\Xi; \Delta; \Gamma \vdash \mathsf{pack}\langle G', t'\rangle \text{ as } \exists X.G'' \preceq \mathsf{pack}\langle G', t'\rangle \text{ as } \exists X.G'' : \exists X.G''$$

By induction hypotheses we already know that $\Xi; \Delta; \Gamma \vdash t' \preceq t' : G''[G'/X]$. But the result follows directly by Prop 10.3 (Compatibility of pack).

Case (**unpack**). Then $t = \mathsf{unpack}\langle X, x\rangle = t_1 \text{ in } t_2$, and therefore:

$$\text{(Eunpack)}\frac{\Xi; \Delta; \Gamma \vdash t_1 : \exists X.G_1 \qquad \Xi; \Delta, X; \Gamma, x : G_1 \vdash t_2 : G_2 \qquad \Xi; \Delta \vdash G_2}{\Xi; \Delta; \Gamma \vdash \mathsf{unpack}\langle X, x\rangle = t_1 \text{ in } t_2 : G_2}$$

where $G = G_2$. Then we have to prove that:

$$\Xi; \Delta; \Gamma \vdash \mathsf{unpack}\langle X, x\rangle = t_1 \text{ in } t_2 \preceq \mathsf{unpack}\langle X, x\rangle = t_1 \text{ in } t_2 : G_2$$

By induction hypotheses we already know that $\Xi; \Delta; \Gamma \vdash t_1 \preceq t_1 : \exists X.G_1$ and $\Xi; \Delta, X; \Gamma, x : G_1 \vdash t_2 \preceq t_2 : G_2$. But the result follows directly by Prop 10.4 (Compatibility of unpack).

□

Definition 10.1 (Operators over evidence).

$$\pi_i^*(\varepsilon) \triangleq \langle E_*, E_*\rangle \quad \text{where } E_* = lift_\Xi(unlift(\pi_i(\varepsilon))) \qquad \pi_i^2(\varepsilon) \triangleq \langle E_*, E_*\rangle \quad \text{where } E_* = \pi_i(\varepsilon)$$

$$\langle E_1, E_2\rangle\,[X] = \langle E_1[X], E_2[X]\rangle \qquad \langle E_1, E_2\rangle[E_3, E_4] = \langle E_1[E_3], E_2[E_4]\rangle$$

$$\langle E_1, E_2\rangle\,[E_3, E_4, X] = \langle E_1[E_3/X], E_2[E_4/X]\rangle$$

Proposition 10.2 (**Compatibility-Epacku**). *If* $\Xi; \Delta; \Gamma \vdash v_{11} \preceq v_{12} : G''[G'/X]$, $\Xi; \Delta \vdash G'$ *and* $\varepsilon \Vdash \Xi; \Delta \vdash \exists X.G'' \sim G$, *then*

$$\Xi; \Delta; \Gamma \vdash \varepsilon(\mathit{packu}\langle G', v_{11}\rangle \text{ as } \exists X.G'') :: G \preceq \varepsilon(\mathit{packu}\langle G', v_{12}\rangle \text{ as } \exists X.G'') :: G : G$$

Proof. First, we are required to prove that

$$\Xi; \Delta; \Gamma \vdash \varepsilon(\text{packu}\langle G', v_{1i}\rangle \text{ as } \exists X.G'') :: G : G$$

But by unfolding the premises we know that $\Xi; \Delta; \Gamma \vdash v_{1i} : G''[G'/X]$, therefore:

$$\text{(Epack \& Easc)}\frac{\Xi; \Delta; \Gamma \vdash v_{1i} : G''[G'/X] \qquad \Xi; \Delta \vdash G' \qquad \varepsilon \Vdash \Xi; \Delta \vdash \exists X.G'' \sim G}{\Xi; \Delta; \Gamma \vdash \varepsilon(\text{pack}\langle G', v_{1i}\rangle \text{ as } \exists X.G'') :: G : G}$$

Consider arbitrary W, ρ, γ such that $W \in \mathcal{S}[\![\Xi]\!]$, $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$ and $(W, \gamma) \in \mathcal{G}_\rho[\![\Gamma]\!]$. We are required to show that

$$(W, \rho(\gamma_1(\varepsilon(\text{packu}\langle G', v_{11}\rangle \text{ as } \exists X.G'') :: G)), \rho(\gamma_2(\varepsilon(\text{packu}\langle G', v_{12}\rangle \text{ as } \exists X.G'') :: G))) \in \mathcal{T}_\rho[\![G]\!]$$

First we have to prove that:

$$W.\Xi_i \vdash \rho(\gamma_i(\varepsilon(\text{packu}\langle G', v_{1i}\rangle \text{ as } \exists X.G'') :: G)) : \rho(G)$$

As we know that $\Xi; \Delta; \Gamma \vdash \varepsilon(\text{packu}\langle G', v_{1i}\rangle \text{ as } \exists X.G'') :: G : G$, by Lemma 6.25 the result follows immediately.

By definition of substitutions

$$\rho(\gamma_i(\varepsilon(\text{packu}\langle G', v_{1i}\rangle \text{ as } \exists X.G'') :: G)) = \varepsilon_i^\rho(\text{packu}\langle \rho(G'), \rho(\gamma_i(v_{1i}))\rangle \text{ as } \exists X.\rho(G'')) :: \rho(G)$$

where $\varepsilon_i^\rho = \rho_i(\varepsilon)$ and $\varepsilon_i^\rho.n = k$. Therefore we have to prove that

$$(W, \varepsilon_1^\rho(\text{packu}\langle \rho(G'), \rho(\gamma_1(v_{11}))\rangle \text{ as } \exists X.\rho(G'')) :: \rho(G), \varepsilon_2^\rho(\text{packu}\langle \rho(G'), \rho(\gamma_2(v_{12}))\rangle \text{ as } \exists X.\rho(G'')) :: \rho(G)) \in \mathcal{T}_\rho[\![G]\!]$$

Or what is the same

$$(W, \varepsilon_1^\rho(\text{packu}\langle \rho(G'), \rho(\gamma_1(v_{11}))\rangle \text{ as } \exists X.\rho(G'')) :: \rho(G), \varepsilon_2^\rho(\text{packu}\langle \rho(G'), \rho(\gamma_2(v_{12}))\rangle \text{ as } \exists X.\rho(G'')) :: \rho(G)) \in \mathcal{V}_\rho[\![G]\!]$$

The type G can be $\exists X.G_1'$, for some G_1', ? or a TypeName.

Let $u_i = \text{packu}\langle \rho(G'), v_{1i}\rangle$ as $\exists X.\rho(G'')$ and $G^* = \exists X.G''$, we have to prove that:

$$(W', \varepsilon_1^\rho u_1 :: \rho(G), \varepsilon_2^\rho u_2 :: \rho(G)) \in \mathcal{V}_\rho[\![G]\!]$$

(1) If $G = \exists X.G_1'$, by the definition of $\mathcal{V}_\rho[\![\exists X.G_1']\!]$, we have to prove that $\forall W'' \geq W, \alpha.\exists R \in \text{Rel}_{W''.j}[\rho(G'), \rho(G')]$ such that $\forall \varepsilon' \Vdash \Xi; dom(\rho) \vdash \exists X.G_1' \sim \exists X.G_1'$ $(\varepsilon'.n = l)$ it is true that

$$(W^*, (\rho_1(\varepsilon) \fatsemi \rho_1(\varepsilon'))[\hat{G}', \hat{\alpha}]v_{11} :: \rho(G_1')[\alpha/X], (\rho_2(\varepsilon) \fatsemi \rho_2(\varepsilon'))[\hat{G}_2, \hat{\alpha}]v_{12} :: \rho(G_1')[\alpha/X]) \in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G_1']\!]$$

where $W^* = W'' \boxtimes (\alpha, \rho(G'), \rho(G'), R)$.

or what is the same, we have to prove that

$$(W^*, (\rho_1(\varepsilon \mathbin{⨟} \varepsilon'))[\hat{G}', \hat{\alpha}]v_{11} :: \rho(G_1')[\alpha/X], (\rho_2(\varepsilon \mathbin{⨟} \varepsilon'))[\hat{G}_2, \hat{\alpha}]v_{12} :: \rho(G_1')[\alpha/X])) \in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G_1']\!]$$

By Proposition 10.8 (decomposition of the evidence) we know that

$$\rho_i(\varepsilon \mathbin{⨟} \varepsilon')[\hat{G}', \hat{\alpha}] = \pi_1^*(\rho_i(\varepsilon \mathbin{⨟} \varepsilon'))[\hat{G}', \hat{\alpha}] \mathbin{⨟} \rho_i(\varepsilon \mathbin{⨟} \varepsilon')[\hat{\alpha}, \hat{\alpha}]$$

Lets take $R = \mathcal{V}_\rho[\![G']\!]$.
Note that

- $W^* = W'' \boxtimes (\alpha, \rho(G'), \rho(G'), \mathcal{V}_\rho[\![G']\!]) \geq W'$
- $E_i' = lift_{W^*.\Xi_i}(\rho(G'))$,
- $E_{i*} = lift_{W^*.\Xi_i}(G_{pi})$, $G_{pi} = unlift(\pi_1(\rho_i(\varepsilon \mathbin{⨟} \varepsilon'))) \sqsubseteq \rho(G'')$,
- $\rho' = \rho[X \mapsto \alpha]$,
- ${\varepsilon_i}^{-1} = \pi_1^*(\rho_i(\varepsilon \mathbin{⨟} \varepsilon'))[\hat{G}', \hat{\alpha}] = \langle E_{i*}[E_i'/X], E_{i*}[\alpha^{E_i}/X]\rangle$, such that ${\varepsilon_i}^{-1} \Vdash W^*.\Xi_i \vdash \rho(G''[G'/X]) \sim \rho(G''[\alpha/X])$, $\alpha^{E_i'} = lift_{W^*.\Xi_i}(\alpha)$, and $E_i' = lift_{W^*.\Xi_i}(\rho(G'))$, ${\varepsilon_i}^{-1}.n = k$ and
- $(W', v_{11}, v_{12}) \in \mathcal{V}_\rho[\![G''[G'/X]]\!]$, then $(W^*, v_{11}, v_{12}) \in \mathcal{V}_\rho[\![G''[G'/X]]\!]$.

By the Lemma §10.6 (compositionality) we know that

$$(W^*, \pi_1(\rho_1(\varepsilon \mathbin{⨟} \varepsilon'))[\hat{G}', \hat{\alpha}]v_{11} :: \rho'(G''), \pi_1(\rho_2(\varepsilon \mathbin{⨟} \varepsilon'))[\hat{G}', \hat{\alpha}]v_{12} :: \rho'(G''))) \in \mathcal{T}_{\rho'}[\![G'']\!]$$

or what is the same

$$(W^*, \pi_1^*(\rho_1(\varepsilon \mathbin{⨟} \varepsilon'))[\hat{G}', \hat{\alpha}]v_{11} :: \rho(G'')[\alpha/X], \pi_1^*(\rho_2(\varepsilon \mathbin{⨟} \varepsilon'))[\hat{G}', \hat{\alpha}]v_{12} :: \rho(G'')[\alpha/X])) \in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G'']\!]$$

Then we know that

$$(\downarrow_k W^*, \varepsilon_1' u_1' :: \rho(G'')[\alpha/X], \varepsilon_2' u_2' :: \rho(G'')[\alpha/X])) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![G'']\!]$$

where $v_{1i} = \varepsilon_{1i}' u_i :: \rho(G''[G'/X])$ and $\varepsilon_i' = \varepsilon_{1i}' \mathbin{⨟} \pi_1^*(\rho_i(\varepsilon \mathbin{⨟} \varepsilon'))[\hat{G}', \hat{\alpha}]$.
Note now that

- $(\downarrow_k W^*, \varepsilon_1' u_1' :: \rho(G'')[\alpha/X], \varepsilon_2' u_2' :: \rho(G'')[\alpha/X])) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![G'']\!]$,
- $(\varepsilon \mathbin{⨟} \varepsilon')[X] \Vdash \Xi; \Delta, X \vdash G'' \sim G_1'$, $(\varepsilon \mathbin{⨟} \varepsilon')[X].n = l$
- $\downarrow_k W^* \in \mathcal{S}[\![\Xi]\!]$ and $(\downarrow_k W^*, \rho') \in \mathcal{D}[\![\Delta, X]\!]$,

Then, by Lemma 10.5 (Ascription Lemma), we know that

$$(\downarrow_{k+l} W^*, (\varepsilon_1' \mathbin{⨟} \rho_1'((\varepsilon \mathbin{⨟} \varepsilon')[X]))u_1' :: \rho'(G_1'), (\varepsilon_2' \mathbin{⨟} \rho_2'((\varepsilon \mathbin{⨟} \varepsilon')[X]))u_2' :: \rho'(G_1')) \in \mathcal{V}_{\rho'}[\![G_1']\!]$$

or what is the same

$$(\downarrow_{k+l} W^*, (\varepsilon_1' \mathbin{⨟} \rho_1(\varepsilon \mathbin{⨟} \varepsilon')[\hat{\alpha}, \hat{\alpha}])u_1' :: \rho(G_1')[\alpha/X], (\varepsilon_2' \mathbin{⨟} \rho_2(\varepsilon \mathbin{⨟} \varepsilon')[\hat{\alpha}, \hat{\alpha}])u_2' :: \rho(G_1')[\alpha/X])) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![G_1']\!]$$

The result follows immediately.

$$(W^*, (\rho_1(\varepsilon \mathbin{⨟} \varepsilon'))[\hat{G}', \hat{\alpha}]v_{11} :: \rho(G_1')[\alpha/X], (\rho_2(\varepsilon \mathbin{⨟} \varepsilon'))[\hat{G}_2, \hat{\alpha}]v_{12} :: \rho(G_1')[\alpha/X])) \in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G_1']\!]$$

(2) If $G \in$ TypeName then $\varepsilon = \langle H_3, \alpha^{E_4}\rangle$. Notice that as α^{E_4} cannot have free type variables therefore H_3 neither. Then $\varepsilon = \rho_i(\varepsilon)$. As α is sync, then let us call $G''' = W.\Xi_i(\alpha)$. We have to prove that:

$$(W, \langle H_3, \alpha^{E_4}\rangle\, u_1 :: \alpha, \langle H_3, \alpha^{E_4}\rangle\, u_2 :: \alpha) \in \mathcal{V}_\rho[\![\alpha]\!]$$

which, by definition of $\mathcal{V}_\rho[\![\alpha]\!]$, is equivalent to prove that:

$$(\downarrow W, \langle H_3, E_4\rangle\, u_1 :: G''', \langle E_3, E_4\rangle\, u_2 :: G''') \in \mathcal{V}_\rho[\![G''']\!]$$

Then we proceed by case analysis on ε:

- (Case $\varepsilon = \langle H_3, \alpha^{\beta^{E_4}}\rangle$). We know that $\langle H_3, \alpha^{\beta^{E_4}}\rangle \vdash \Xi;\Delta \vdash G^* \sim \alpha$, then by Lemma 6.29, $\langle H_3, \beta^{E_4}\rangle \vdash \Xi;\Delta \vdash G^* \sim G'''$. As $\beta^{E_4} \sqsubseteq G'''$, then G''' can either be ? or β.
If $G''' = ?$, then by definition of $\mathcal{V}_\rho[\![?]\!]$, we have to prove that the resulting values belong to $\mathcal{V}_\rho[\![\beta]\!]$. Also as $\langle H_3, \beta^{E_4}\rangle \vdash \Xi;\Delta \vdash G^* \sim ?$, by Lemma 6.27, $\langle H_3, \beta^{E_4}\rangle \vdash \Xi;\Delta \vdash G^* \sim \beta$, and then we proceed just like this case once again (this is process is finite as there are no circular references by construction and it ends up in something different to a type name).
If $G''' = \beta$ we use an analogous argument as for $G'' = ?$.
- (Case $\varepsilon = \langle H_3, \alpha^{H_4}\rangle$). We have to prove that

$$(\downarrow W, \langle H_3, H_4\rangle\, u_1 :: G''', \langle H_3, H_4\rangle\, u_2 :: G''') \in \mathcal{V}_\rho[\![G''']\!]$$

By Lemma 6.29, $\langle H_3, H_4\rangle \vdash \Xi;\Delta \vdash G^* \sim G''$. Then if $G'' = ?$, we proceed as the case $G' = ?$, with the evidence $\varepsilon = \langle H_3, H_4\rangle$. If $G'' \in \textsc{HeadType}$, we proceed as the previous case where $G' = \forall X.G$, and the evidence $\varepsilon = \langle H_3, H_4\rangle$.
Also, we have to prove that $(\forall \Xi', \varepsilon', G_1^*$, such that $\varepsilon'.n = k$, $\varepsilon' = \langle \alpha^{E_1^{**}}, E_2^{**}\rangle$ $(\downarrow W \in \mathcal{S}[\![\Xi']\!] \land \varepsilon' \vdash \Xi' \vdash \alpha \sim G_1^*)$, we get that

$$(\downarrow_1 W, \varepsilon'(\langle H_3, \alpha^{H_4}\rangle u_1 :: \alpha) :: G_1^*, \varepsilon'(\langle H_4, \alpha^{E_{22}}\rangle u_2 :: \alpha) :: G_1^*) \in \mathcal{T}_\rho[\![G_1^*]\!])$$

or what is the same $((\langle H_3, \alpha^{H_4}\rangle \mathbin{⨾} \varepsilon')$ fails the result follows immediately)

$$(\downarrow_{1+k} W, (\langle H_3, \alpha^{H_4}\rangle \mathbin{⨾} \varepsilon')u_1 :: G_1^*, (\langle H_2, \alpha^{H_4}\rangle \mathbin{⨾} \varepsilon')u_2 :: G_1^*) \in \mathcal{V}_\rho[\![G_1^*]\!])$$

By definition of transitivity and Lemma 6.30, we know that

$$\langle H_3, \alpha^{H_4}\rangle \mathbin{⨾} \langle \alpha^{E_1^{**}}, E_2^{**}\rangle = \langle H_3, H_4\rangle \mathbin{⨾} \langle E_1^{**}, E_2^{**}\rangle$$

We know that $\langle E_1^{**}, E_2^{**}\rangle \vdash \Xi' \vdash G'' \sim G_1^*$. Since $\langle E_1^{**}, E_2^{**}\rangle \vdash \Xi \vdash G'' \sim G_1^*$, $\downarrow_1 W \in \mathcal{S}[\![\Xi']\!]$, $(\downarrow_1 W, \langle H_3, H_4\rangle u_1 :: G'', \langle H_1, H_4\rangle u_2 :: G'') \in \mathcal{V}_\rho[\![G'']\!]$, by Lemma 6.17, we know that (since $(\langle H_3, \alpha^{H_4}\rangle \mathbin{⨾} \varepsilon')$ does not fail then $(\langle H_3, H_4\rangle \mathbin{⨾} \langle E_1^{**}, E_2^{**}\rangle)$ also does not fail by the transitivity rules)

$$(\downarrow_{1+k} W, (\langle H_3, H_4\rangle \mathbin{⨾} \langle E_1^{**}, E_2^{**}\rangle)u_1 :: G_1^*, (\langle H_3, H_4\rangle \mathbin{⨾} \langle E_1^{**}, E_2^{**}\rangle)u_2 :: G_1^*) \in \mathcal{V}_\rho[\![G_1^*]\!])$$

The result follows immediately.

(3) If $G = ?$ we have the following cases:
- ($G = ?$, $\varepsilon = \langle H_3, H_4\rangle$). By the definition of $\mathcal{V}_\rho[\![?]\!]$ in this case we have to prove that:

$$(W, \rho_1(\varepsilon)u_1 :: \rho(G), \rho_2(\varepsilon)u_2 :: \rho(G)) \in \mathcal{V}_\rho[\![const(H_4)]\!]$$

but as $const(H_4) = \exists X.?$, we proceed just like the case where $G = \exists X.G_1'$, where $G_1' = ?$.
- ($G = ?$, $\varepsilon = \langle H_3, \alpha^{E_4}\rangle$). Notice that as α^{E_4} cannot have free type variables therefore E_3 neither. Then $\varepsilon = \rho_i(\varepsilon)$. By the definition of $\mathcal{V}_\rho[\![?]\!]$ we have to prove that

$$(W, \langle H_3, \alpha^{E_4}\rangle\, u_1 :: \alpha, \langle H_3, \alpha^{E_4}\rangle\, u_2 :: \alpha) \in \mathcal{V}_\rho[\![\alpha]\!]$$

Note that by Lemma 6.27 we know that $\varepsilon \vdash \Xi;\Delta \vdash G^* \sim \alpha$. Then we proceed just like the case $G \in \textsc{TypeName}$.

□

Proposition 10.3 (**Compatibility-Epack**). *If* $\Xi;\Delta;\Gamma \vdash t_1 \preceq t_2 : G''[G'/X]$, $\Xi;\Delta \vdash G'$, *then*

$$\Xi;\Delta;\Gamma \vdash pack\langle G', t_1\rangle \text{ as } \exists X.G'' \preceq pack\langle G', t_2\rangle \text{ as } \exists X.G'' : \exists X.G''$$

Proof. First, we are required to prove that

$$\Xi;\Delta;\Gamma \vdash \mathsf{pack}\langle G', t_i\rangle \text{ as } \exists X.G'' : \exists X.G''$$

But by unfolding the premises we know that $\Xi;\Delta;\Gamma \vdash t_i : G''[G'/X]$, therefore:

$$\text{(Epack \& Easc)}\frac{\Xi;\Delta;\Gamma \vdash t_i : G''[G'/X] \qquad \Xi;\Delta \vdash G'}{\Xi;\Delta;\Gamma \vdash \mathsf{pack}\langle G', t_i\rangle \text{ as } \exists X.G'' : \exists X.G''}$$

Consider arbitrary W, ρ, γ such that $W \in \mathcal{S}[\![\Xi]\!]$, $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$ and $(W, \gamma) \in \mathcal{G}_\rho[\![\Gamma]\!]$. We are required to show that

$$(W, \rho(\gamma_1(\mathsf{pack}\langle G', t_1\rangle \text{ as } \exists X.G'')), \rho(\gamma_2(\mathsf{pack}\langle G', t_2\rangle \text{ as } \exists X.G''))) \in \mathcal{T}_\rho[\![\exists X.G'']\!]$$

First we have to prove that:

$$W.\Xi_i \vdash \rho(\gamma_i((\mathsf{pack}\langle G', t_i\rangle \text{ as } \exists X.G''))) : \rho(\exists X.G'')$$

As we know that $\Xi;\Delta;\Gamma \vdash (\mathsf{pack}\langle G', t_i\rangle \text{ as } \exists X.G'') : \exists X.G''$, by Lemma 6.25 the result follows immediately.

By definition of substitutions

$$\rho(\gamma_i((\mathsf{pack}\langle G', t_i\rangle \text{ as } \exists X.G''))) = (\mathsf{pack}\langle \rho(G'), \rho(\gamma_i(t_i))\rangle \text{ as } \exists X.\rho(G''))$$

Therefore we have to prove that

$$(W, (\mathsf{pack}\langle \rho(G'), \rho(\gamma_1(t_1))\rangle \text{ as } \exists X.\rho(G'')), (\mathsf{pack}\langle \rho(G'), \rho(\gamma_2(t_2))\rangle \text{ as } \exists X.\rho(G''))) \in \mathcal{T}_\rho[\![\exists X.G'']\!]$$

Second, consider arbitrary $i < W.j, \Xi_1$. Either there exist v_1 such that:

$$W.\Xi_1 \triangleright (\mathsf{pack}\langle \rho(G'), \rho(\gamma_1(t_1))\rangle \text{ as } \exists X.\rho(G'')) \longmapsto^i \Xi_1 \triangleright v_1$$

or

$$W.\Xi_1 \triangleright (\mathsf{pack}\langle \rho(G'), \rho(\gamma_1(t_1))\rangle \text{ as } \exists X.\rho(G'')) \longmapsto^i \mathbf{error}$$

Let us suppose that $W.\Xi_1 \triangleright (\mathsf{pack}\langle \rho(G'), \rho(\gamma_1(t_1))\rangle \text{ as } \exists X.\rho(G'')) \longmapsto^i \Xi_1 \triangleright v_1$. Hence, by inspection of the operational semantics, it follows that there exist $i_1 \leq i$, Ξ_{11} and v_{11} such that:

$$W.\Xi_1 \triangleright (\mathsf{pack}\langle \rho(G'), \rho(\gamma_1(t_1))\rangle \text{ as } \exists X.\rho(G'')) \longmapsto^{i_1} \Xi_{11} \triangleright (\mathsf{pack}\langle \rho(G'), v_{11}\rangle \text{ as } \exists X.\rho(G'')) \longmapsto^1$$
$$\Xi_{11} \triangleright \varepsilon_1^\rho(\mathsf{packu}\langle \rho(G'), v_{11}\rangle \text{ as } \exists X.\rho(G'')) :: \exists X.\rho(G'')$$

where $\varepsilon = \langle \exists X.G'', \exists X.G''\rangle$ and $\varepsilon_i^\rho = \rho_i(\varepsilon)$.

We instantiate the hypothesis $\Xi;\Delta;\Gamma \vdash t_1 \preceq t_2 : G''[G'/X]$ with W, ρ and γ to obtain that:

$$(W, \rho(\gamma_1(t_1)), \rho(\gamma_2(t_2))) \in \mathcal{T}_\rho[\![G''[G'/X]]\!]$$

We instantiate $\mathcal{T}_\rho[\![G''[G'/X]]\!]$ with i_1, Ξ_{11} and v_{11} (note that $i_1 \leq i < W.j$), hence there exists v_{12} and W_1, such that $W_1 \geq W$, $W_1.j = W.j - i_1$, $W.\Xi_2 \triangleright \rho(\gamma_2(t_2)) \longmapsto^* W'.\Xi_2 \triangleright v_{12}$, $W'.\Xi_1 = \Xi_{11}$, and $(W_1, v_{11}, v_{12}) \in \mathcal{V}_\rho[\![G''[G'/X]]\!]$ (Note that if $W.\Xi_1 \triangleright \rho(\gamma_1(t_1)) \longmapsto^{i_1}$ **error** the result follows immediately). Let' s take $W' = \downarrow_1 W_1$. Note that we get that $(W', v_{11}, v_{12}) \in \mathcal{V}_\rho[\![G''[G'/X]]\!]$.

Then we have to prove that

$$(W', \varepsilon_1^\rho(\mathsf{packu}\langle \rho(G'), v_{11}\rangle \text{ as } \exists X.\rho(G'')) :: \exists X.\rho(G''),$$
$$\varepsilon_2^\rho(\mathsf{packu}\langle \rho(G'), v_{12}\rangle \text{ as } \exists X.\rho(G'')) :: \exists X.\rho(G'')) \in \mathcal{V}_\rho[\![\exists X.\rho(G'')]\!]$$

Let $u_i = \text{packu}\langle \rho(G'), v_{1i}\rangle$ as $\exists X.\rho(G'')$ and $\exists X.G_1 = \exists X.G''$, we have to prove that:

$$(W', \varepsilon_1^{\rho} u_1 :: \exists X.\rho(G''), \varepsilon_2^{\rho} u_2 :: \exists X.\rho(G'')) \in \mathcal{V}_\rho[\![\exists X.G'']\!]$$

(1) By the definition of $\mathcal{V}_\rho[\![\exists X.G_1']\!]$, we have to prove that $\forall W'' \geq W', \alpha.\exists R \in \text{REL}_{W''.j}[\rho(G'), \rho(G')]$ such that $\forall \varepsilon' \Vdash \Xi; dom(\rho) \vdash \exists X.G_1' \sim \exists X.G_1'$ ($\varepsilon'.n = l$) it is true that

$$(W^*, (\rho_1(\varepsilon) \,⨾\, \rho_1(\varepsilon'))[\hat{G}', \hat{\alpha}]v_{11} :: \rho(G_1')[\alpha/X], (\rho_2(\varepsilon) \,⨾\, \rho_2(\varepsilon'))[\hat{G}_2, \hat{\alpha}]v_{12} :: \rho(G_1')[\alpha/X]) \in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G_1']\!]$$

where $W^* = W'' \boxtimes (\alpha, \rho(G'), \rho(G'), R)$.
or what is the same, we have to prove that

$$(W^*, (\rho_1(\varepsilon \,⨾\, \varepsilon'))[\hat{G}', \hat{\alpha}]v_{11} :: \rho(G_1')[\alpha/X], (\rho_2(\varepsilon \,⨾\, \varepsilon'))[\hat{G}_2, \hat{\alpha}]v_{12} :: \rho(G_1')[\alpha/X]) \in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G_1']\!]$$

By Proposition 10.8 (decomposition of the evidence) we know that

$$\rho_i(\varepsilon \,⨾\, \varepsilon')[\hat{G}', \hat{\alpha}] = \pi_1^*(\rho_i(\varepsilon \,⨾\, \varepsilon'))[\hat{G}', \hat{\alpha}] \,⨾\, \rho_i(\varepsilon \,⨾\, \varepsilon')[\hat{\alpha}, \hat{\alpha}]$$

Lets take $R = \mathcal{V}_\rho[\![G']\!]$.
Note that

- $W^* = W'' \boxtimes (\alpha, \rho(G'), \rho(G'), \mathcal{V}_\rho[\![G']\!]) \geq W'$
- $E_i' = lift_{W^*.\Xi_i}(\rho(G'))$,
- $E_{i*} = lift_{W^*.\Xi_i}(G_{pi})$, $G_{pi} = unlift(\pi_1(\rho_i(\varepsilon \,⨾\, \varepsilon'))) \sqsubseteq \rho(G'')$,
- $\rho' = \rho[X \mapsto \alpha]$,
- $\varepsilon_i{}^{-1} = \pi_1^*(\rho_i(\varepsilon \,⨾\, \varepsilon'))[\hat{G}', \hat{\alpha}] = \langle E_{i*}[E_i'/X], E_{i*}[\alpha^{E_i}/X]\rangle$, such that $\varepsilon_i{}^{-1} \Vdash W^*.\Xi_i \vdash \rho(G''[G'/X]) \sim \rho(G''[\alpha/X])$, $\alpha^{E_i'} = lift_{W^*.\Xi_i}(\alpha)$, and $E_i' = lift_{W^*.\Xi_i}(\rho(G'))$, $\varepsilon_i{}^{-1}.n = k$ and
- $(W', v_{11}, v_{12}) \in \mathcal{V}_\rho[\![G''[G'/X]]\!]$, then $(W^*, v_{11}, v_{12}) \in \mathcal{V}_\rho[\![G''[G'/X]]\!]$.

By the Lemma §10.6 (compositionality) we know that

$$(W^*, \pi_1(\rho_1(\varepsilon \,⨾\, \varepsilon'))[\hat{G}', \hat{\alpha}]v_{11} :: \rho'(G''), \pi_1(\rho_2(\varepsilon \,⨾\, \varepsilon'))[\hat{G}', \hat{\alpha}]v_{12} :: \rho'(G'')) \in \mathcal{T}_{\rho'}[\![G'']\!]$$

or what is the same

$$(W^*, \pi_1^*(\rho_1(\varepsilon \,⨾\, \varepsilon'))[\hat{G}', \hat{\alpha}]v_{11} :: \rho(G'')[\alpha/X], \pi_1^*(\rho_2(\varepsilon \,⨾\, \varepsilon'))[\hat{G}', \hat{\alpha}]v_{12} :: \rho(G'')[\alpha/X]) \in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G'']\!]$$

Then we know that

$$(\downarrow_k W^*, \varepsilon_1' u_1' :: \rho(G'')[\alpha/X], \varepsilon_2' u_2' :: \rho(G'')[\alpha/X]) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![G'']\!]$$

where $v_{1i} = \varepsilon_{1i}' u_i :: \rho(G''[G'/X])$ and $\varepsilon_i' = \varepsilon_{1i}' \,⨾\, \pi_1^*(\rho_i(\varepsilon \,⨾\, \varepsilon'))[\hat{G}', \hat{\alpha}]$.
Note now that

- $(\downarrow_k W^*, \varepsilon_1' u_1' :: \rho(G'')[\alpha/X], \varepsilon_2' u_2' :: \rho(G'')[\alpha/X]) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![G'']\!]$,
- $(\varepsilon \,⨾\, \varepsilon')[X] \Vdash \Xi; \Delta, X \vdash G'' \sim G_1'$, $(\varepsilon \,⨾\, \varepsilon')[X].n = l$
- $\downarrow_k W^* \in \mathcal{S}[\![\Xi]\!]$ and $(\downarrow_k W^*, \rho') \in \mathcal{D}[\![\Delta, X]\!]$,

Then, by Lemma 10.5 (Ascription Lemma), we know that

$$(\downarrow_{k+l} W^*, (\varepsilon_1' \,⨾\, \rho_1'((\varepsilon \,⨾\, \varepsilon')[X]))u_1' :: \rho'(G_1'), (\varepsilon_2' \,⨾\, \rho_2'((\varepsilon \,⨾\, \varepsilon')[X]))u_2' :: \rho'(G_1')) \in \mathcal{V}_{\rho'}[\![G_1']\!]$$

or what is the same

$$(\downarrow_{k+l} W^*, (\varepsilon_1' \,⨾\, \rho_1(\varepsilon \,⨾\, \varepsilon')[\hat{\alpha}, \hat{\alpha}])u_1' :: \rho(G_1')[\alpha/X], (\varepsilon_2' \,⨾\, \rho_2(\varepsilon \,⨾\, \varepsilon')[\hat{\alpha}, \hat{\alpha}])u_2' :: \rho(G_1')[\alpha/X]) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![G_1']\!]$$

The result follows immediately.

$$(W^*, (\rho_1(\varepsilon \,⨾\, \varepsilon'))[\hat{G}', \hat{\alpha}]v_{11} :: \rho(G_1')[\alpha/X], (\rho_2(\varepsilon \,⨾\, \varepsilon'))[\hat{G}_2, \hat{\alpha}]v_{12} :: \rho(G_1')[\alpha/X]) \in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G_1']\!]$$

□

Proposition 10.4 (**Compatibility-Eunpack**). *If* $\Xi; \Delta; \Gamma \vdash t_1 \preceq t_2 : \exists X.G_1$, $\Xi; \Delta, X; \Gamma, x : G_1 \vdash t'_1 \preceq t'_2 : G_2$ *and* $\Xi; \Delta \vdash G_2$, *then* $\Xi; \Delta; \Gamma \vdash unpack\langle X, x\rangle = t_1 \; in \; t'_1 \preceq unpack\langle X, x\rangle = t_2 \; in \; t'_2 : G_2$.

Proof. First, we are required to prove that

$$\Xi; \Delta; \Gamma \vdash \mathsf{unpack}\langle X, x\rangle = t_i \;\mathsf{in}\; t'_i : G_2$$

But by unfolding the premises we know that $\Xi; \Delta; \Gamma \vdash t_i : \exists X.G_1$, $\Xi; \Delta, X; \Gamma, x : G_1 \vdash t'_i : G_2$ and $\Xi; \Delta \vdash G_2$, therefore:

$$\text{(Eunpack)}\frac{\Xi; \Delta; \Gamma \vdash t_i : \exists X.G_1 \qquad \Xi; \Delta, X; \Gamma, x : G_1 \vdash t'_i : G_2 \qquad \Xi; \Delta \vdash G_2}{\Xi; \Delta; \Gamma \vdash \mathsf{unpack}\langle X, x\rangle = t_i \;\mathsf{in}\; t'_i : G_2}$$

Consider arbitrary W, ρ, γ such that $W \in \mathcal{S}[\![\Xi]\!]$, $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$ and $(W, \gamma) \in \mathcal{G}_\rho[\![\Gamma]\!]$. We are required to show that

$$(W, \rho(\gamma_1(\mathsf{unpack}\langle X, x\rangle = t_1 \;\mathsf{in}\; t'_1)), \rho(\gamma_2(\mathsf{unpack}\langle X, x\rangle = t_2 \;\mathsf{in}\; t'_2))) \in \mathcal{T}_\rho[\![G_2]\!]$$

First we have to prove that:

$$W.\Xi_i \vdash \rho(\gamma_i(\mathsf{unpack}\langle X, x\rangle = t_i \;\mathsf{in}\; t'_i)) : \rho(G_2)$$

As we know that $\Xi; \Delta; \Gamma \vdash \mathsf{unpack}\langle X, x\rangle = t_i \;\mathsf{in}\; t'_i : G_2$, by Lemma 6.25 the result follows immediately.

By definition of substitutions

$$\rho(\gamma_i(\mathsf{unpack}\langle X, x\rangle = t_i \;\mathsf{in}\; t'_i)) = \mathsf{unpack}\langle X, x\rangle = \rho(\gamma_i(t_i)) \;\mathsf{in}\; \rho(\gamma_i(t'_i))$$

Therefore we have to prove that

$$(W, \mathsf{unpack}\langle X, x\rangle = \rho(\gamma_1(t_1)) \;\mathsf{in}\; \rho(\gamma_1(t'_1)), \mathsf{unpack}\langle X, x\rangle = \rho(\gamma_2(t_2)) \;\mathsf{in}\; \rho(\gamma_2(t'_2))) \in \mathcal{T}_\rho[\![G_2]\!]$$

Second, consider arbitrary $i < W.j, \Xi_1$. Either there exist v_1 such that:

$$W.\Xi_1 \triangleright \mathsf{unpack}\langle X, x\rangle = \rho(\gamma_1(t_1)) \;\mathsf{in}\; \rho(\gamma_1(t'_1)) \longmapsto^i \Xi_1 \triangleright v_1$$

or

$$W.\Xi_1 \triangleright \mathsf{unpack}\langle X, x\rangle = \rho(\gamma_1(t_1)) \;\mathsf{in}\; \rho(\gamma_1(t'_1)) \longmapsto^i \Xi_1 \triangleright \mathbf{error}$$

Let us suppose that $W.\Xi_1 \triangleright \mathsf{unpack}\langle X, x\rangle = \rho(\gamma_1(t_1)) \;\mathsf{in}\; \rho(\gamma_1(t'_1)) \longmapsto^i \Xi_1 \triangleright v_1$.

Hence, by inspection of the operational semantics, it follows that there exist $i_1 \leq i$, Ξ_{11} and v_{11} such that:

$$W.\Xi_1 \triangleright \rho(\gamma_1(t_1)) \longmapsto^{i_1} \Xi_{11} \triangleright v_{11}$$

Instantiate the second conjunct of $\Xi; \Delta; \Gamma \vdash t_1 \preceq t_2 : \exists X.G_1$ with W, ρ, and γ. Note that $W \in \mathcal{S}[\![\Xi]\!]$, $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$ and $(W, \gamma) \in \mathcal{G}_\rho[\![\Gamma]\!]$. Then we have that $(W, \rho(\gamma_1(t_1)), \rho(\gamma_2(t_2))) \in \mathcal{T}_\rho[\![\exists X.G_1]\!]$. Instantiate this with i_1, Ξ_{11} and v_{11} . Note that $i_1 < W.j$ which follows from $i_1 \leq i < W.j$.

Hence,there exists $W_1 \geq W$ and v_{12} such that $W.\Xi_2 \triangleright \rho(\gamma_2(t_2)) \longmapsto^* W_1.\Xi_2 \triangleright v_{12}$, $(W_1, v_{11}, v_{12}) \in \mathcal{V}_\rho[\![\exists X.G_1]\!]$ and $W_1.j + i_1 = W.j$.

Hence, $v_{1i} = \varepsilon'_i(\mathsf{packu}\langle G'_i, v'_i\rangle \;\mathsf{as}\; \exists X.G''_i) :: \exists X.\rho(G_1)$, where $\varepsilon'_1 = k.n$ and $v'_i = \varepsilon_{pi} u_i :: G_{pi}$.

From $(W_1, v_{11}, v_{12}) \in \mathcal{V}_\rho[\![\exists X.G_1]\!]$, it follows that there exists $R \in \mathrm{Rel}_{W_1.j}[G'_1, G'_2]$ such that $\forall \varepsilon' \Vdash \Xi; \Delta \vdash \exists X.G_1 \sim \exists X.G_1$ $(\varepsilon'.n = l)$ it is true that

$$(W'_1, (\varepsilon'_1 \mathbin{;} \rho_1(\varepsilon'))[\hat{G}'_1, \hat{\alpha}]v'_1 :: \rho(G_1)[\alpha/X], (\varepsilon'_2 \mathbin{;} \rho_2(\varepsilon'))[\hat{G}'_2, \hat{\alpha}]v'_2 :: \rho(G_1)[\alpha/X]) \in \mathcal{T}_{\rho[X \mapsto \alpha]}[\![G_1]\!]$$

where $W_1' = W_1 \boxtimes (\alpha, G_1', G_2', R)$. If we take $\varepsilon' = \mathcal{I}_\Xi(\exists X.G_1, \exists X.G_1)$, then

$$(\varepsilon_i' \fatsemi \rho_i(\varepsilon')) = \varepsilon_i'$$

Therefore we know that

$$(W_1', \varepsilon_1'[\hat{G}_1', \hat{\alpha}]v_1' :: \rho(G_1)[\alpha/X], \varepsilon_2'[\hat{G}_2', \hat{\alpha}]v_2' :: \rho(G_1)[\alpha/X]) \in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G_1]\!]$$

If $W_1'.\Xi_1 \triangleright \varepsilon_1'[\hat{G}_1', \hat{\alpha}]v_1' :: \rho(G_1)[\alpha/X] \longmapsto$ **error** the result follows immediately. Otherwise, if

$$W_1'.\Xi_1 \triangleright \varepsilon_1'[\hat{G}_1', \hat{\alpha}]v_1' :: \rho(G_1)[\alpha/X] \longmapsto^{k+l} W_1'.\Xi_1 \triangleright v_{p1}$$

where $v_{p1} = (\varepsilon_{p1} \fatsemi \varepsilon_1'[\hat{G}_1', \hat{\alpha}]u_1 :: \rho(G_1)[\alpha/X]$, then

$$W_1'.\Xi_2 \triangleright \varepsilon_2'[\hat{G}_2', \hat{\alpha}]v_2' :: \rho(G_1)[\alpha/X] \longmapsto^* W_1'.\Xi_2 \triangleright v_{p2}$$

where $v_{p2} = (\varepsilon_{p2} \fatsemi \varepsilon_2'[\hat{G}_2', \hat{\alpha}]u_2 :: \rho(G_1)[\alpha/X]$ and $(W_2', v_{p1}, v_{p2}) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![G_1]\!]$, where $W_2' =\downarrow_{k+l} W_1'$ and $W_2'.j + k + l = W_1'.j$.

Note that

$$W.\Xi_1 \triangleright \mathsf{unpack}\langle X, x\rangle = \rho(\gamma_1(t_1)) \text{ in } \rho(\gamma_1(t_1')) \longmapsto^{i_1}$$
$$W_1.\Xi_1 \triangleright \mathsf{unpack}\langle X, x\rangle = v_{11} \text{ in } \rho(\gamma_1(t_1')) \longmapsto^{k+l} W_1.\Xi_1 \triangleright t_2[\alpha/X][v_{p1}/x] \longmapsto^{i_2} \Xi_1 \triangleright v_1$$

where $i = i_1 + k + l + i_2$.

Instantiate the second conjunct of $\Xi; \Delta, X; \Gamma, x : G_1 \vdash t_1' \preceq t_2' : G_2$ with W_2', $\rho[X \mapsto \alpha]$, $\gamma[x \mapsto (v_{p1}, v_{p2})]$. Note that $W_2' \in \mathcal{S}[\![\Xi]\!](W_2' \geq W)$, $(W_2', \rho[X \mapsto \alpha]) \in \mathcal{D}[\![\Delta, X]\!]$ and $(W_2', \gamma[x \mapsto (v_{p1}, v_{p2})]) \in \mathcal{G}_\rho[\![\Gamma, x : G_1]\!]$. Then we have that

$$(W_2', \gamma_1(\rho(t_1'))[\hat{\alpha}/X][v_{p1}/x], \gamma_2(\rho(t_2'))[\hat{\alpha}/X][v_{p2}/x]) \in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G_2]\!]$$

Instantiate this with $i_2 < W_2'.j = W.j - i_1 - k - l (i_2 = i - i_1 - k - l, i < W.j)$, Ξ_1 and v_1 . Hence, there exists $W_2 \geq W_2'$ and v_2 such that

$W'.\Xi_2 \triangleright \gamma_2(\rho(t_2'))[\hat{\alpha}/X][v_{p2}/x] \longmapsto^* W_2.\Xi_2 \triangleright v_2$, $W_2.\Xi_1 = \Xi_1$, $W_2.j + i_2 = W_2'.j$ and

$$(W_2, v_1, v_2) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![G_2]\!]$$

We are required to show that there exists $W_2 \geq W$ and v_2, such that

$$W.\Xi_2 \triangleright \mathsf{unpack}\langle X, x\rangle = \rho(\gamma_2(t_2)) \text{ in } \rho(\gamma_2(t_2')) \longmapsto^* W_2.\Xi_2 \triangleright v_2$$

, $W_2.j + i = W.j (W_2.j = W.j - i_1 - k - l - i_2, i = i_1 + k + l + i_2)$ and $(W_2, v_1, v_2) \in \mathcal{V}_\rho[\![G_2]\!]$, which follows from $(W_2, v_1, v_2) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![G_2]\!]$ and $\Xi; \Delta \vdash G_2$. ◻

Proposition 10.5 (**Ascriptions Preserve Relations**). *If* $(W, v_1, v_2) \in \mathcal{V}_\rho[\![G]\!]$, $\varepsilon \Vdash \Xi; \Delta \vdash G \sim G'$, $W \in \mathcal{S}[\![\Xi]\!]$ *and* $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$, *then* $(W, \rho_1(\varepsilon)v_1 :: \rho(G'), \rho_2(\varepsilon)v_2 :: \rho(G')) \in \mathcal{T}_\rho[\![G']\!]$.

Proof. We only prove the case for existential, the other cases are in 6.2.

Case ($G = \exists X.G_1''$ and $G' = \exists X.G_1'$). We know that

$$(W, v_1, v_2) \in \mathcal{V}_\rho[\![\exists X.G_1'']\!]$$

Where $v_i = \varepsilon_i(\mathsf{packu}\langle G_i{}^*, v_i'\rangle \text{ as } \exists X.\rho(G_i''')) :: \exists X.\rho(G_1'')$ and $\varepsilon_i \vdash W.\Xi_i \vdash \exists X.\rho(G_i''') \sim \exists X.\rho(G_1'')$. Let's suppose that $\rho_1(\varepsilon).n = k$ and $\varepsilon_1.n = m$. We have to prove that

$$(W, \rho_1(\varepsilon)v_1 :: \exists X.\rho(G_1'), \rho_2(\varepsilon)v_2 :: \exists X.\rho(G_1')) \in \mathcal{T}_\rho[\![\exists X.G_1']\!]$$

If $(\varepsilon_1 \fatsemi \rho_1(\varepsilon))$ fails, then we apply Lemma 6.26 to show that $(\varepsilon_2 \fatsemi \rho_2(\varepsilon))$ also fails, therefore the proof holds immediately. In the other case, $(\varepsilon_i \fatsemi \rho_i(\varepsilon))$ do not fail, then by the definition of $\mathcal{T}_\rho[\![\exists X.G_1']\!]$, we have to prove that:

$$(\downarrow_k W, (\varepsilon_1 \mathbin{;} \rho_1(\varepsilon))(\text{packu}\langle G_1{}^*, v_1'\rangle \text{ as } \exists X.\rho(G_1''')) :: \exists X.\rho(G_1'), (\varepsilon_2 \mathbin{;} \rho_2(\varepsilon))(\text{packu}\langle G_2{}^*, v_2'\rangle \text{ as } \exists X.\rho(G_2''')) :: \exists X.\rho(G_1')) \in \mathcal{V}_\rho[\![\exists X.G_1']\!]$$

or what is the same:

$\forall W'' \geq \downarrow_k W, \alpha. \exists R \in \text{Rel}_{W''.j}[G_1^*, G_2^*]$.
$(W''.\Xi_1 \vdash G_1^* \wedge W''.\Xi_2 \vdash G_2^* \wedge \forall \Xi, \varepsilon' \Vdash \Xi; dom(\rho) \vdash \exists X.G_1' \sim \exists X.G_1', \Xi \in \mathcal{S}[\![\Xi]\!], \varepsilon'.n = l$.
$(W''', (\varepsilon_1 \mathbin{;} \rho_1(\varepsilon \mathbin{;} \varepsilon'))[\hat{G_1^*}, \hat{\alpha}]v_1' :: \rho(G_1')[\alpha/X], (\varepsilon_2 \mathbin{;} \rho_2(\varepsilon \mathbin{;} \varepsilon'))[\hat{G_2^*}, \hat{\alpha}]v_2' :: \rho(G_1')[\alpha/X]) \in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G_1']\!]$

where $W''' = ((W'') \boxtimes (\alpha, G_1^*, G_2^*, R))$.

Let's suppose that $v_i' = \varepsilon_i{}^* u_i :: G_i'''[G_i^*]$. Therefore, we are required to prove that

$$((\downarrow_{k+l+m} W''')(\varepsilon_1{}^* \mathbin{;} \varepsilon_1 \mathbin{;} \rho_1(\varepsilon \mathbin{;} \varepsilon'))[\hat{G_1^*}, \hat{\alpha}]u_1 :: \rho(G_1')[\alpha/X],$$
$$\varepsilon_2{}^* \mathbin{;} (\varepsilon_2 \mathbin{;} \rho_2(\varepsilon \mathbin{;} \varepsilon'))[\hat{G_2^*}, \hat{\alpha}]u_2 :: \rho(G_1')[\alpha/X]) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![G_1']\!]$$

Note that by Lemma 10.9 we get that

$$(\varepsilon_i \mathbin{;} \rho_i(\varepsilon \mathbin{;} \varepsilon'))[\hat{G_i^*}, \hat{\alpha}] = (\varepsilon_i \mathbin{;} \pi_1^2(\rho_i(\varepsilon \mathbin{;} \varepsilon')))[\hat{G_i^*}, \hat{\alpha}] \mathbin{;} \rho_i(\varepsilon \mathbin{;} \varepsilon')[\hat{\alpha}, \hat{\alpha}] =$$
$$(\varepsilon_i \mathbin{;} \rho_i(\pi_1^2(\varepsilon \mathbin{;} \varepsilon')))[\hat{G_i^*}, \hat{\alpha}] \mathbin{;} \rho_i(\varepsilon \mathbin{;} \varepsilon')[\hat{\alpha}, \hat{\alpha}]$$

By premise, we know that $(W, v_1, v_2) \in \mathcal{V}_\rho[\![\exists X.G_1'']\!]$. Then, we instantiate this definition with $(\uparrow_k W'') \geq W$ $(W'' \geq (\downarrow_k W) \Rightarrow (\uparrow_k W'') \geq \uparrow_k\downarrow_k W)$ and α. Therefore, $\exists R \in \text{Rel}_{W''.j}[G_1^*, G_2^*]$, such that for all evidence $\varepsilon'' \Vdash \Xi'; dom(\rho) \vdash \exists X.G_1'' \sim \exists X.G_1''$, in particular $\varepsilon'' = \pi_1^2(\varepsilon \mathbin{;} \varepsilon')$ $(\pi_1^2(\varepsilon \mathbin{;} \varepsilon').n = k)$. Therefore, we know that $(W''' = (W'' \boxtimes (\alpha, G_1^*, G_2^*, R)))$:

$$(W''', (\varepsilon_1 \mathbin{;} \rho_1(\pi_1^2(\varepsilon \mathbin{;} \varepsilon')))[\hat{G_1^*}, \hat{\alpha}]v_1' :: \rho(G_1'')[\alpha/X], (\varepsilon_2 \mathbin{;} \rho_2(\pi_1^2(\varepsilon \mathbin{;} \varepsilon'))[\hat{G_2^*}, \hat{\alpha}]v_2' :: \rho(G_1'')[\alpha/X]) \in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G_1'']\!]$$

Then, we get that:

$$((\downarrow_{k+l} W'''), v_1''', v_2''') \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![G_1'']\!]$$

where $v_i''' = \varepsilon_i{}^* \mathbin{;} (\varepsilon_i \mathbin{;} \rho_i(\pi_1^2(\varepsilon \mathbin{;} \varepsilon')))[\hat{G_i^*}, \hat{\alpha}]u_i :: \rho(G_1'')[\alpha/X]$.

By induction hypothesis on $((\downarrow_{k+l} W'''), v_1''', v_2''') \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![G_1'']\!]$, with $(\varepsilon \mathbin{;} \varepsilon')[X] \Vdash \Xi; \Delta, X \vdash G_1'' \sim G_1'$ $((\varepsilon \mathbin{;} \varepsilon')[X].n = m)$, $(\downarrow_{k+l} W''') \in \mathcal{S}[\![\Xi]\!]$ and $((\downarrow_{k+l} W'''), \rho') \in \mathcal{D}[\![\Delta, X]\!]$, $\rho' = \rho[X \mapsto \alpha]$, we get that:

$$((\downarrow_{k+l} W'''), \rho_1'((\varepsilon \mathbin{;} \varepsilon')[X])v_1''' :: \rho(G_1')[\alpha/X], \rho_2'((\varepsilon \mathbin{;} \varepsilon')[X])v_2''' :: \rho(G_1')[\alpha/X]) \in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G_1']\!]$$

or what is the same (note that $\rho_i'((\varepsilon \mathbin{;} \varepsilon')[X]) = \rho_i(\varepsilon \mathbin{;} \varepsilon')[\hat{\alpha}, \hat{\alpha}]$):

$$((\downarrow_{k+l} W'''), \rho_1(\varepsilon \mathbin{;} \varepsilon')[\hat{\alpha}, \hat{\alpha}]v_1''' :: \rho(G_1')[\alpha/X], \rho_2(\varepsilon \mathbin{;} \varepsilon')[\hat{\alpha}, \hat{\alpha}]v_2''' :: \rho(G_1')[\alpha/X]) \in \mathcal{T}_{\rho[X\mapsto\alpha]}[\![G_1']\!]$$

or what is the same:

$$((\downarrow_{k+l+m} W'''), v_1^*, v_2^*) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![G_1']\!]$$

where $v_i^* = \varepsilon_i{}^* \mathbin{;} ((\varepsilon_i \mathbin{;} \rho_i(\pi_1^2(\varepsilon \mathbin{;} \varepsilon')))[\hat{G_i^*}, \hat{\alpha}] \mathbin{;} \rho_i(\varepsilon \mathbin{;} \varepsilon')[\hat{\alpha}, \hat{\alpha}])u_i :: \rho(G_1')[\alpha/X]$.

By the reduction rule

$$W'''.\Xi_1 \triangleright (\varepsilon_1 \mathbin{;} \rho_1(\varepsilon \mathbin{;} \varepsilon'))[\hat{G_1^*}, \hat{\alpha}]v_1' :: \rho(G_1')[\alpha/X] \longrightarrow^{k+m+l} W'''.\Xi_1 \triangleright v_1^*$$

Therefore, the results follows immediately $(((\downarrow_{k+l+m} W'''), v_1^*, v_2^*) \in \mathcal{V}_{\rho[X\mapsto\alpha]}[\![G_1']\!])$.

□

Proposition 10.6 (**CompositionalityEx**). *If*

- $W.\Xi_i(\alpha) = \rho(G')$ *and* $W.\kappa(\alpha) = \mathcal{V}_\rho[\![G']\!]$,
- $E'_i = lift_{W.\Xi_i}(\rho(G'))$,
- $E_i = lift_{W.\Xi_i}(G_p)$ *for some* $G_p \sqsubseteq \rho(G)$,
- $\rho' = \rho[X \mapsto \alpha]$,
- $\varepsilon_i = \langle E_i[\alpha^{E'_i}/X], E_i[E'_i/X]\rangle$, *such that* $\varepsilon_i \vdash W.\Xi_i \vdash \rho(G[\alpha/X]) \sim \rho(G[G'/X])$, *and*
- ${\varepsilon_i}^{-1} = \langle E_i[E'_i/X], E_i[\alpha^{E'_i}/X]\rangle$, *such that* ${\varepsilon_i}^{-1} \vdash W.\Xi_i \vdash \rho(G[G'/X]) \sim \rho(G[\alpha/X])$, *then*

(1)

$$(W, \varepsilon'_1 u_1 :: \rho'(G), \varepsilon'_2 u_2 :: \rho'(G)) \in \mathcal{V}_{\rho'}[\![G]\!] \Rightarrow$$
$$(W, \varepsilon_1(\varepsilon'_1 u_1 :: \rho(G)) :: \rho(G\,[G'/X]), \varepsilon_2(\varepsilon'_2 u_2 :: \rho(G)) :: \rho(G\,[G'/X])) \in \mathcal{T}_\rho[\![G\,[G'/X]]\!]$$

(2)

$$(W, \varepsilon'_1 u_1 :: \rho(G\,[G'/X]), \varepsilon'_2 u_2 :: \rho(G\,[G'/X])) \in \mathcal{V}_\rho[\![G\,[G'/X]]\!] \Rightarrow$$
$$(W, {\varepsilon_1}^{-1}(\varepsilon'_1 u_1 :: \rho(G\,[G'/X])) :: \rho'(G), {\varepsilon_2}^{-1}(\varepsilon'_2 u_2 :: \rho(G\,[G'/X])) :: \rho'(G)) \in \mathcal{T}_{\rho'}[\![G]\!]$$

Proof. We only prove the case for existential, the other cases are in 6.2. We proceed by induction on G. Let $v_i = \varepsilon'_i u_i :: \rho'(G)$, $\Delta = dom(\rho)$. We prove (1) first. Let's suppose that $\varepsilon'_1.n = k$, $\varepsilon_1.n = l$ and ${\varepsilon_1}^{-1}.n = m$.

Case $(\exists Y.G_1)$. We know that

$$(W, \varepsilon'_1 u_1 :: \rho'(G), \varepsilon'_2 u_2 :: \rho'(G)) \in \mathcal{V}_{\rho'}[\![G]\!]$$

where $u_i = \text{packu}\langle G^*_i, v'_i\rangle$ as $\exists Y.G''_i$ and $G = \exists Y.G_1$. Therefore, we have to prove that

$$(W, \varepsilon_1(\varepsilon'_1 u_1 :: \rho(G)) :: \rho(G\,[G'/X]), \varepsilon_2(\varepsilon'_2 u_2 :: \rho(G)) :: \rho(G\,[G'/X])) \in \mathcal{V}_\rho[\![G\,[G'/X]]\!]$$

If $\varepsilon'_i \,\mathring{;}\, \varepsilon_i$ is not defined, the result follows immediately. If it is defined, we have to prove that:

$$((\downarrow_l W), (\varepsilon'_1 \,\mathring{;}\, \varepsilon_1)u_1 :: \rho(G\,[G'/X], (\varepsilon'_2 \,\mathring{;}\, \varepsilon_2)u_2 :: \rho(G\,[G'/X])) \in \mathcal{V}_\rho[\![G\,[G'/X]]\!]$$

or what is the same by the definition of $\mathcal{V}_\rho[\![G\,[G'/X]]\!]$, we have to prove that:

$$\begin{array}{l}\forall W'' \geq (\downarrow_l W), \beta.\exists R \in \text{Rel}_{W''.j}[G^*_1, G^*_2].\\ (W''.\Xi_1 \vdash G^*_1 \wedge W''.\Xi_2 \vdash G^*_2 \wedge \forall \varepsilon' \Vdash \Xi; \Delta \vdash \exists Y.G_1[G'/X] \sim \exists Y.G_1[G'/X] \wedge \varepsilon'.n = k'\\ (W''', (\varepsilon'_1 \,\mathring{;}\, \varepsilon_1 \,\mathring{;}\, \rho_1(\varepsilon'))[\hat{G}^*_1, \hat{\beta}]v'_1 :: \rho(G_1[G'/X][\beta/Y]), (\varepsilon'_2 \,\mathring{;}\, \varepsilon_2 \,\mathring{;}\, \rho_2(\varepsilon'))[\hat{G}^*_2, \hat{\beta}]v'_2 :: \rho(G_1[G'/X][\beta/Y]))\\ \qquad \in \mathcal{T}_{\rho[Y\mapsto\beta]}[\![G_1[G'/X]]\!]\end{array}$$

where $W''' = ((W'') \boxtimes (\alpha, G^*_1, G^*_2, R))$. Therefore, we are required to prove that

$$((\downarrow_{k+l+k'} W''')({\varepsilon_1}^* \,\mathring{;}\, (\varepsilon'_1 \,\mathring{;}\, \varepsilon_1 \,\mathring{;}\, \rho_1(\varepsilon'))[\hat{G}^*_1, \hat{\beta}])u'_1 :: \rho(G_1[G'/X][\beta/Y]),$$
$$({\varepsilon_2}^* \,\mathring{;}\, (\varepsilon'_2 \,\mathring{;}\, \varepsilon_2 \,\mathring{;}\, \rho_2(\varepsilon'))[\hat{G}^*_2, \hat{\beta}])u'_2 :: \rho(G_1[G'/X][\beta/Y])) \in \mathcal{T}_{\rho[Y\mapsto\beta]}[\![G_1[G'/X]]\!]$$

where $v'_i = {\varepsilon_i}^* u'_i :: G''_i[G^*_i/Y]$.

Note that by Lemma 10.10 we know that $\varepsilon_i = \rho_i(\varepsilon^{**})[\alpha, \rho(G'), X]$ for some $\varepsilon^{**} \Vdash \Xi; \Delta, X \vdash \exists Y.G_1 \sim \exists Y.G_1$. Therefore, by Lemma 10.11 we get that for some $\varepsilon^* \Vdash \Xi; \Delta, X \vdash \exists Y.G_1 \sim \exists Y.G_1$:

$$(\varepsilon'_i \,\mathring{;}\, \varepsilon_i \,\mathring{;}\, \rho_i(\varepsilon'))[\hat{G}^*_i, \hat{\beta}] = (\varepsilon'_i \,\mathring{;}\, \rho_i(\varepsilon^{**})[\alpha, \rho(G'), X] \,\mathring{;}\, \rho_i(\varepsilon'))[\hat{G}^*_i, \hat{\beta}] =$$
$$(\varepsilon'_i \,\mathring{;}\, \rho_i(\varepsilon^*)[\alpha, \alpha, X])[\hat{G}^*_i, \hat{\beta}] \,\mathring{;}\, (\pi^*_2(\rho_i(\varepsilon^*))[\alpha, \rho(G'), Y] \,\mathring{;}\, \rho_i(\varepsilon'))[\hat{\beta}, \hat{\beta}] =$$

$$(\varepsilon_i' \fatsemi \rho_i'(\varepsilon^*))[\hat{G}_i^*, \hat{\beta}] \fatsemi (\pi_2^*(\rho_i(\varepsilon^*))[\alpha, \rho(G'), Y] \fatsemi \rho_i(\varepsilon'))[\hat{\beta}, \hat{\beta}]$$

By premise, we know that $(W, \varepsilon_1' u_1 :: \rho'(G), \varepsilon_2' u_2 :: \rho'(G)) \in \mathcal{V}_{\rho'}[\![\exists Y.G_1]\!]$. Then, we instantiate this definition with $(\uparrow_l W'') \geq W$ $(W'' \geq (\downarrow_l W) \Rightarrow \uparrow W'' \geq (\uparrow_l \downarrow_l W))$ and β. Therefore, $\exists R \in \text{REL}_{W''.j}[G_1^*, G_2^*]$, such that for all evidence $\varepsilon'' \Vdash \Xi; \Delta, X \vdash \exists X.G_1' \sim \exists X.G_1'$, in particular, we instantiate with $\varepsilon'' = \varepsilon^*[X]$ $(\varepsilon''.n = l)$. Therefore, we know that $(W''' = ((W'') \boxtimes (\beta, G_1^*, G_2^*, R)))$:

$$(\uparrow_l W'', (\varepsilon_1' \fatsemi \rho_1'(\varepsilon^*))[\hat{G}_1^*, \hat{\beta}]v_1' :: \rho'(G_1)[\beta/Y], (\varepsilon_2' \fatsemi \rho_2'(\varepsilon^*))[\hat{G}_2^*, \hat{\beta}]v_2' :: \rho'(G_1)[\beta/Y] \in \mathcal{T}_{\rho'[Y\mapsto\beta]}[\![G_1]\!]$$

Therefore, we know that

$$(\downarrow_k W''', v_1'', v_2'') \in \mathcal{V}_{\rho'[Y\mapsto\beta]}[\![G_1]\!]$$

where $v_i'' = \varepsilon_i{}^* \fatsemi (\varepsilon_i' \fatsemi \rho_i'(\varepsilon^*))[\hat{G}_i^*, \hat{\beta}]u_i' :: \rho'(G_1)[\beta/Y]$.

Note that, for some $G_{ph} \sqsubseteq \rho[Y \mapsto \beta](G_1)$, we get $E_i^* = \mathit{lift}_{W'''.\Xi_i}(G_{ph})$ such that:

- $\mathit{unlift}(\pi_2(\rho[Y \mapsto \beta]_i(\varepsilon^*))) = G_{ph} \sqsubseteq \rho[Y \mapsto \beta](G_1)$ and $E_i^* = \mathit{lift}_{W'''.\Xi_i}(G_{ph})$
- $\pi_2^*(\rho_i(\varepsilon^*))[\hat{\beta}, \hat{\beta}] = \pi_2^*(\rho[Y \mapsto \beta]_i(\varepsilon^*)) = \langle E_i^*, E_i^* \rangle$, by the definition of $\pi_2^*()[.]$
- $\pi_2^*(\rho_i(\varepsilon^*))[\alpha, \rho(G'), X][\hat{\beta}, \hat{\beta}] = \pi_2^*(\rho[Y \mapsto \beta]_i(\varepsilon^*))[\alpha, \rho(G'), X]$
- $\langle E_i^*[\alpha^{E_i'}/X], E_i^*[E_i'/X] \rangle = \langle E_i^*, E_i^* \rangle\, [\alpha, \rho(G'), X] = \pi_2^*(\rho_i(\varepsilon^*))[\alpha, \rho(G'), X][\hat{\beta}, \hat{\beta}]$

Now, by the induction hypothesis we get:

- $(\downarrow_k W''', v_1'', v_2'') \in \mathcal{V}_{\rho'[Y\mapsto\beta]}[\![G_1]\!]$
- $W_i'''.\Xi(\alpha) = \rho[Y \mapsto \beta](G')$ and $W.\kappa(\alpha) = \mathcal{V}_{\rho[Y\mapsto\beta]}[\![G']\!]$,
- $E_i' = \mathit{lift}_{W'''.\Xi_i}(\rho[Y \mapsto \beta](G'))$,
- $E_i^* = \mathit{lift}_{W'''.\Xi_i}(G_{ph})$, $G_{ph} \sqsubseteq \rho[Y \mapsto \beta](G_1)$,
- $\rho'' = \rho[Y \mapsto \beta][X \mapsto \alpha]$,
- $\varepsilon_{ih} = \langle E_i^*[\alpha^{E_i'}/X], E_i^*[E_i'/X] \rangle = \pi_2^*(\rho_i(\varepsilon^*))[\alpha, \rho(G'), Y][\hat{\beta}, \hat{\beta}]$ $(\varepsilon_{ih}.n = l)$, such that

$$\varepsilon_{ih} \vdash W'''.\Xi_i \vdash \rho[Y \mapsto \beta](G_1[\alpha/X]) \sim \rho[Y \mapsto \beta](G_1[G'/X])$$

$$(\downarrow_k W''', \varepsilon_{1h} v_1'' :: \rho[Y \mapsto \beta](G_1[G'/X]), \varepsilon_{2h} v_2'' :: \rho[Y \mapsto \beta](G_1[G'/X])) \in \mathcal{T}_{\rho[Y\mapsto\beta]}[\![G_1[G/X]]\!]$$

If the combination of evidence does not succeed, then the result follows immediately. Otherwise, we get that

$$(\downarrow_{k+l} W''' W''', v_1''', v_2''') \in \mathcal{V}_{\rho[Y\mapsto\beta]}[\![G_1[G/X]]\!]$$

where $v_i''' = (\varepsilon_i{}^* \fatsemi (\varepsilon_i' \fatsemi \rho_i'(\varepsilon^*))[\hat{G}_i^*, \hat{\beta}] \fatsemi \varepsilon_{ih})u_i' :: \rho[Y \mapsto \beta](G_1[G'/X])$

By the ascription Lemma 10.5:

- $(\downarrow_{k+l} W''', v_1''', v_2''') \in \mathcal{V}_{\rho[Y\mapsto\beta]}[\![G_1[G/X]]\!]$
- $\varepsilon'[Y] \Vdash \Xi; \Delta, Y \vdash G_1[G'/X] \sim G_1[G'/X]$ $(\varepsilon'[Y].n = k')$
- $\downarrow_{k+l} W''' \in \mathcal{S}[\![\Xi]\!]$ and $(\downarrow_{k+l} W''', \rho[Y \mapsto \beta]) \in \mathcal{D}[\![\Delta, Y]\!]$

then we have:

$$(\downarrow_{k+l} W''', \rho_1(\varepsilon'))[\hat{\beta}, \hat{\beta}]v_1''' :: \rho[Y \mapsto \beta](G_1[G'/X]),$$
$$\rho_2(\varepsilon'))[\hat{\beta}, \hat{\beta}]v_2''' :: \rho[Y \mapsto \beta](G_1[G'/X])) \in \mathcal{T}_{\rho[Y\mapsto\beta]}[\![G_1[G/X]]\!]$$

If the combination of evidence does not succeed, then the result follows immediately. Otherwise, we get that

$$(\downarrow_{k+l+k'} W''', v_1'''', v_2'''') \in \mathcal{V}_{\rho[Y\mapsto\beta]}[\![G_1[G/X]]\!]$$

where $v_i'''' = (\varepsilon_i{}^* \fatsemi (\varepsilon_i' \fatsemi \rho_i'(\varepsilon^*))[\hat{G}_i^*, \hat{\beta}] \fatsemi \varepsilon_{ih} \fatsemi \rho_i(\varepsilon'))[\hat{\beta}, \hat{\beta}])u_i' :: \rho[Y \mapsto \beta](G_1[G'/X])$ Note that

$$W'''.\Xi_1 \triangleright (\varepsilon_1' \fatsemi \varepsilon_1 \fatsemi \rho_1(\varepsilon'))[\hat{G}_1^*, \hat{\beta}]v_1' :: \rho(G_1[G'/X][\beta/Y]) \longrightarrow^{k+l+k'} W'''.\Xi_1 \triangleright v_1''''$$

And, we have to prove

$$(W''', (\varepsilon_1' \mathbin{;} \varepsilon_1 \mathbin{;} \rho_1(\varepsilon'))[\hat{G}_1^*, \hat{\beta}]v_1' :: \rho(G_1[G'/X][\beta/Y]), (\varepsilon_2' \mathbin{;} \varepsilon_2 \mathbin{;} \rho_2(\varepsilon'))[\hat{G}_2^*, \hat{\beta}]v_2' :: \rho(G_1[G'/X][\beta/Y])) \in \mathcal{T}_{\rho[Y\mapsto\beta]}[\![G_1[G'/X]]\!]$$

Therefore, the result follows immediately $(((\downarrow_{k+l+k'} W'''), v_1'''', v_2'''') \in \mathcal{V}_{\rho[Y\mapsto\beta]}[\![G_1[G/X]]\!])$.

□

Lemma 10.7. *If* $\varepsilon \Vdash \Xi; \Delta \vdash \exists X.G_1 \sim \exists X.G_2$ *then* $\varepsilon\ [X] \Vdash \Xi; \Delta, X \vdash G_1 \sim G_2$.

Proof. Straightforward by induction on the evidences. □

Lemma 10.8.

$$\varepsilon[E_1, E_2] = \pi_1^*(\varepsilon)[E_1, E_2] \mathbin{;} \varepsilon[E_2, E_2] = \pi_1^2(\varepsilon)[E_1, E_2] \mathbin{;} \varepsilon[E_2, E_2]$$

Proof. Straightforward induction on the evidence structure. □

Lemma 10.9.

$$(\varepsilon \mathbin{;} \varepsilon')[E_1, E_2] = (\varepsilon \mathbin{;} \pi_1^*(\varepsilon'))[E_1, E_2] \mathbin{;} \varepsilon'[E_2, E_2] = (\varepsilon \mathbin{;} \pi_1^2(\varepsilon'))[E_1, E_2] \mathbin{;} \varepsilon'[E_2, E_2]$$

Proof. Straightforward induction on the evidence structure. □

Lemma 10.10. *If* $\varepsilon_i \Vdash W.\Xi_i \vdash \rho(G) \sim \rho(G)$, $W \in \mathcal{S}[\![\Xi]\!]$ *and* $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$, *then* $\exists \varepsilon \Vdash \Xi, \Delta \vdash G \sim G$ *such that* $\varepsilon_i = \rho_i(\varepsilon)$.

Proof. Straightforward induction on the evidence structure. □

Lemma 10.11 (**Evidence decomposition**). *If*

- $\varepsilon_1 \Vdash \Xi; \Delta, X, Y \vdash G \sim G$
- $\varepsilon_2 \Vdash \Xi; \Delta, X \vdash G[G'/Y] \sim G''$ *and* $\Xi; \Delta \vdash G'$
- $W \in \mathcal{S}[\![\Xi]\!]$, $(W, \rho[X \mapsto \alpha][Y \mapsto \beta]) \in \mathcal{D}[\![\Delta, X, Y]\!]$, $W.\Xi_i(\alpha) = \rho(G_i)$ *and* $W.\Xi_i(\beta) = \rho(G')$

then $\exists \varepsilon \Vdash \Xi; \Delta, X, Y \vdash G \sim G$

$$(\rho_i(\varepsilon_1)[\beta, G', Y] \mathbin{;} \rho_i(\varepsilon_2))[G_i, \alpha, X] = (\rho_i(\varepsilon)[\beta, \beta, Y])[G_i, \alpha, X] \mathbin{;} (\pi_2^*(\rho_i(\varepsilon))[\beta, G, Y] \mathbin{;} \rho_i(\varepsilon_2))[\alpha, \alpha, X]$$

Proof. We proceed by induction on G.

Case ($G = B$ and $G'' = B$). Then, we know that $\varepsilon_i = \langle B, B\rangle$. Therefore, if we choose $\varepsilon = \langle B, B\rangle$ the results follows immediately.

Case ($G = G_1'' \to G_2''$, and $G'' = G_1' \to G_2'$). We know that

- $\varepsilon_1 \Vdash \Xi; \Delta, X, Y \vdash G_1'' \to G_2'' \sim G_1'' \to G_2''$ implies that

$$idom^\sharp(\varepsilon_1) \Vdash \Xi; \Delta, X, Y \vdash G_1'' \sim G_1''$$

- $\varepsilon_2 \Vdash \Xi; \Delta, X \vdash (G_1'' \to G_2'')[G'/Y] \sim G_1' \to G_2'$ implies that

$$idom^\sharp(\varepsilon_2) \Vdash \Xi; \Delta, X \vdash G_1''[G'/Y] \sim G_1'$$

Therefore by the induction hypothesis, we know that $\exists \varepsilon' \Vdash \Xi; \Delta, X, Y \vdash G_1'' \sim G_1''$ such that

$$(\rho_i(idom^\sharp(\varepsilon_1))[\beta, G', Y] \mathbin{;} \rho_i(idom^\sharp(\varepsilon_2)))[G_i, \alpha, X] =$$
$$(\rho_i(\varepsilon')[\beta, \beta, Y])[G_i, \alpha, X] \mathbin{;} (\pi_2^*(\rho_i(\varepsilon'))[\beta, G, Y] \mathbin{;} \rho_i(idom^\sharp(\varepsilon_2)))[\alpha, \alpha, X]$$

Also we know that

- $\varepsilon_1 \Vdash \Xi; \Delta, X, Y \vdash G_1'' \rightarrow G_2'' \sim G_1'' \rightarrow G_2''$ implies that
$$icod^\sharp(\varepsilon_1) \Vdash \Xi; \Delta, X, Y \vdash G_2'' \sim G_2''$$
- $\varepsilon_2 \Vdash \Xi; \Delta, X \vdash (G_1'' \rightarrow G_2'')[G'/Y] \sim G_1' \rightarrow G_2'$ implies that
$$icod^\sharp(\varepsilon_2) \Vdash \Xi; \Delta, X \vdash G_2''[G'/Y] \sim G_2'$$

Therefore by the induction hypothesis, we know that $\exists \varepsilon'' \Vdash \Xi; \Delta, X, Y \vdash G_2'' \sim G_2''$ such that
$$(\rho_i(icod^\sharp(\varepsilon_1))[\beta, G', Y] \mathbin{⨾} \rho_i(icod^\sharp(\varepsilon_2)))[G_i, \alpha, X] =$$
$$(\rho_i(\varepsilon'')[\beta, \beta, Y])[G_i, \alpha, X] \mathbin{⨾} (\pi_2^*(\rho_i(\varepsilon''))[\beta, G, Y] \mathbin{⨾} \rho_i(icod^\sharp(\varepsilon_2)))[\alpha, \alpha, X]$$
Therefore, it follows that $\exists \varepsilon \Vdash \Xi; \Delta, X, Y \vdash G_1'' \rightarrow G_2'' \sim G_1'' \rightarrow G_2''$, such that the result follows immediately ($\varepsilon = \langle \pi_1(\varepsilon') \rightarrow \pi_1(\varepsilon''), \pi_2(\varepsilon') \rightarrow \pi_2(\varepsilon'')\rangle$). Note that

- $idom^\sharp(\varepsilon) = \varepsilon'$
- $icod^\sharp(\varepsilon) = \varepsilon''$
- $idom^\sharp((\rho_i(\varepsilon_1)[\beta, G', Y] \mathbin{⨾} \rho_i(\varepsilon_2))[G_i, \alpha, X]) =$
$(\rho_i(idom^\sharp(\varepsilon_1))[\beta, G', Y] \mathbin{⨾} \rho_i(idom^\sharp(\varepsilon_2)))[G_i, \alpha, X] =$
$(\rho_i(\varepsilon')[\beta, \beta, Y])[G_i, \alpha, X] \mathbin{⨾} (\pi_2^*(\rho_i(\varepsilon'))[\beta, G, Y] \mathbin{⨾} \rho_i(idom^\sharp(\varepsilon_2)))[\alpha, \alpha, X] =$
$idom^\sharp((\rho_i(\varepsilon)[\beta, \beta, Y])[G_i, \alpha, X] \mathbin{⨾} (\pi_2^*(\rho_i(\varepsilon))[\beta, G, Y] \mathbin{⨾} \rho_i(\varepsilon_2))[\alpha, \alpha, X])$
- $icod^\sharp((\rho_i(\varepsilon_1)[\beta, G', Y] \mathbin{⨾} \rho_i(\varepsilon_2))[G_i, \alpha, X]) =$
$(icod^\sharp(\rho_i(icod^\sharp(\varepsilon_1))[\beta, G', Y] \mathbin{⨾} \rho_i(icod^\sharp(\varepsilon_2)))[G_i, \alpha, X] =$
$(\rho_i(\varepsilon'')[\beta, \beta, Y])[G_i, \alpha, X] \mathbin{⨾} (\pi_2^*(\rho_i(\varepsilon''))[\beta, G, Y] \mathbin{⨾} \rho_i(icod^\sharp(\varepsilon_2)))[\alpha, \alpha, X] =$
$icod^\sharp((\rho_i(\varepsilon)[\beta, \beta, Y])[G_i, \alpha, X] \mathbin{⨾} (\pi_2^*(\rho_i(\varepsilon))[\beta, G, Y] \mathbin{⨾} \rho_i(\varepsilon_2))[\alpha, \alpha, X])$

Note that two evidences are equals if and only if their $idom^\sharp$ and $icod^\sharp$ equals too.

Case ($G = \forall X.G_1''$ and $G'' = \forall X.G_1'$). Similar to function case.

Case ($G = G_1 \times G2$). Similar to function case.

Case ($G = \alpha$). This means that evidences do not have type variables, therefore, type substitutions are not applied. For this reason, the result follows immediately.

Case ($G = \beta$). This means that evidences do not have type variables, therefore, type substitutions are not applied. For this reason, the result follows immediately.

Case ($G = \beta$). This means that evidences do not have type variables, therefore, type substitutions are not applied. For this reason, the result follows immediately.

Case ($G = X$). Then, we know that $\varepsilon_1 = \langle X, X\rangle$ and $\varepsilon_2 = \langle X, X\rangle$. Therefore, with $\varepsilon = \langle X, X\rangle$ the result follows immediately.

Case ($G = Y$). Then, we know that $\varepsilon_1 = \langle Y, Y\rangle$. Since, $\varepsilon_2 \Vdash \Xi; \Delta, X \vdash G' \sim G''$ and $\Xi; \Delta \vdash G'$ (without X), we know that
$$\rho_i(\varepsilon_2)[G_i, \alpha, X] = \rho_i(\varepsilon_2)[\alpha, \alpha, X] = \rho_i(\varepsilon_2)$$
Therefore, $\exists \varepsilon = \langle Y, Y\rangle$, such that the result follows immediately.

Case ($G = Z$). Then, we know that $\varepsilon_1 = \langle Z, Z\rangle$ and $\varepsilon_2 = \langle Z, Z\rangle$. Therefore, with $\varepsilon = \langle Z, Z\rangle$ the result follows immediately.

Case ($G = ?$). We follow by case in the evidences.

- $\varepsilon_1 = \langle ?, ?\rangle$, then $\exists \varepsilon = \varepsilon_2$ such that the results follows immediately (by Lemma 10.8).

$\boxed{G \leqslant G}$ **Strict type precision**

$$\frac{G_1 \leqslant G_2}{\exists X.G_1 \leqslant \exists X.G_2}$$

$\boxed{\Omega \vdash \Xi_1 \triangleright s : G \leqslant \Xi_2 \triangleright s : G}$ **Strict term precision** (for conciseness, s ranges over both t and u)

$$(\leqslant\text{packu}_\varepsilon)\frac{G_1' \leqslant G_2' \qquad \Omega \vdash \Xi_1 \triangleright v_1 : G_1[G_1'/X] \leqslant \Xi_2 \triangleright v_2 : G_2[G_2'/X] \qquad \exists X.G_1 \sqsubseteq \exists X.G_2}{\Omega \vdash \Xi_1 \triangleright \mathsf{packu}\langle G_1', v_1\rangle \text{ as } \exists X.G_1 : \exists X.G_1 \leqslant \Xi_2 \triangleright \mathsf{packu}\langle G_2', v_2\rangle \text{ as } \exists X.G_2 : \exists X.G_2}$$

$$(\leqslant\text{pack}_\varepsilon)\frac{G_1' \leqslant G_2' \qquad \Omega \vdash \Xi_1 \triangleright t_1 : G_1[G_1'/X] \leqslant \Xi_2 \triangleright t_2 : G_2[G_2'/X] \qquad \exists X.G_1 \leqslant \exists X.G_2}{\Omega \vdash \Xi_1 \triangleright \mathsf{pack}\langle G_1', t_1\rangle \text{ as } \exists X.G_1 : \exists X.G_1 \leqslant \Xi_2 \triangleright \mathsf{pack}\langle G_2', t_2\rangle \text{ as } \exists X.G_2 : \exists X.G_2}$$

$$(\leqslant\text{unpack}_\varepsilon)\frac{\Omega \vdash \Xi_1 \triangleright t_1 : \exists X.G_1 \leqslant \Xi_2 \triangleright t_2 : \exists X.G_2 \qquad \Omega, x : G_1 \sqsubseteq G_2 \vdash \Xi_1 \triangleright t_1' : G_1' \leqslant \Xi_2 \triangleright t_2' : G_2'}{\Omega \vdash \Xi_1 \triangleright \mathsf{unpack}\langle X, x\rangle = t_1 \text{ in } t_1' : G_1' \leqslant \Xi_2 \triangleright \mathsf{unpack}\langle X, x\rangle = t_2 \text{ in } t_2' : G_2'}$$

$\boxed{G \rightharpoonup G}$ **Type matching**

$$? \rightharpoonup \exists X.?$$

$\boxed{\Omega \vdash v : G \leqslant_v v : G}$ **Strict value precision**

$$(\leqslant\text{packu})\frac{G_1' \leqslant G_2' \qquad \Omega \vdash v_1 : G_1'' \leqslant_v v_2 : G_2'' \qquad \exists X.G_1 \sqsubseteq \exists X.G_2 \qquad G_1'' \sqcap G_1[G_1'/X] \leqslant G_2'' \sqcap G_2[G_2'/X]}{\Omega \vdash \mathsf{pack}\langle G_1', v_1\rangle \text{ as } \exists X.G_1 : \exists X.G_1 \leqslant_v \mathsf{pack}\langle G_2', v_2\rangle \text{ as } \exists X.G_2 : \exists X.G_2}$$

$\boxed{\Omega \vdash t : G \leqslant t : G}$ **Strict term precision**

$$(\leqslant\text{pack})\frac{G_1' \leqslant G_2' \qquad \Omega \vdash t_1 : G_1'' \leqslant t_2 : G_2'' \qquad \exists X.G_1 \leqslant \exists X.G_2 \qquad G_1'' \sqcap G_1[G_1'/X] \leqslant G_2'' \sqcap G_2[G_2'/X]}{\Omega \vdash \mathsf{pack}\langle G_1', t_1\rangle \text{ as } \exists X.G_1 : \exists X.G_1 \leqslant \mathsf{pack}\langle G_2', t_2\rangle \text{ as } \exists X.G_2 : \exists X.G_2}$$

$$(\leqslant\text{unpack})\frac{\Omega \vdash t_1 : G_1 \leqslant t_2 : G_2 \qquad \Omega, x : schm_e^\sharp(G_1) \sqsubseteq schm_e^\sharp(G_2) \vdash t_1' : G_1' \leqslant t_2' : G_2'}{\Omega \vdash \mathsf{unpack}\langle X, x\rangle = t_1 \text{ in } t_1' : G_1' \leqslant \mathsf{unpack}\langle X, x\rangle = t_2 \text{ in } t_2' : G_2'}$$

Fig. 27. $\mathsf{GSF}_\varepsilon^\exists$ and $\mathsf{GSF}^\exists$: Strict term precision

- $\varepsilon_2 = \langle ?, ?\rangle$, then $\exists \varepsilon = \varepsilon_1$ such that the results follows immediately (by Lemma 10.8).
- The other evidence cases are covered in other cases of the proof.

□

Proposition 12.5. *If* $\Xi; \Delta; \Gamma \vdash t_1 \approx t_2 : G$, *then* $\Xi; \Delta; \Gamma \vdash t_1 \approx^{ctx} t_2 : G$.

Proof. Similar to Th. 6.32. □

10.5 A Weak Dynamic Gradual Guarantee for $\mathsf{GSF}^\exists$

Proposition 10.12. *If* $\Omega \vdash t_1^* : G_1^* \leqslant t_2^* : G_2^*$, $\Omega \equiv \Gamma_1 \sqsubseteq \Gamma_2$, $\Delta; \Gamma_i \vdash t_i^* \rightsquigarrow t_i^{**} : G_i^*$, *then* $\Omega \vdash \triangleright t_1^{**} : G_1^* \leqslant \Xi_2 \triangleright t_2^{**} : G_2^*$.

Proof. We follow by induction on $\Omega \vdash t_1^* : G_1^* \leqslant t_2^* : G_2^*$. We avoid the notation $\Omega \vdash t_1^* : G_1^* \leqslant t_2^* : G_2^*$, and use $t_1^* \leqslant t_2^*$ instead, for simplicity, when the typing environments are not relevant. We use metavariable v or u in GSF to range over constants, functions and type abstractions. We only proof here the cases related to existential types. Other cases where proved in Section 5.

Case (⩽v).

$$(\leqslant\text{v})\frac{\Omega \vdash u_1 : G_1^* \leqslant_v u_2 : G_2^* \qquad G_1^* \leqslant G_2^*}{\Omega \vdash u_1 : G_1^* \leqslant u_2 : G_2^*}$$

$$(\text{Gu})\frac{\Delta;\Gamma_1 \vdash u_1 \rightsquigarrow u_1' : G_1^* \qquad \varepsilon_{G_1^*} = \mathcal{I}(G_1^*, G_1^*)}{\Delta;\Gamma_1 \vdash u_1 \rightsquigarrow \varepsilon_{G_1^*} u_1' :: G_1^* : G_1^*}$$

$$(\text{Gu})\frac{\Delta;\Gamma_2 \vdash u_2 \rightsquigarrow u_2' : G_2^* \qquad \varepsilon_{G_2^*} = \mathcal{I}(G_2^*, G_2^*)}{\Delta;\Gamma_2 \vdash u_2 \rightsquigarrow \varepsilon_{G_2^*} u_2' :: G_2^* : G_2^*}$$

We have to prove that $\Omega \vdash \varepsilon_{G_1^*} u_1' :: G_1^* \leqslant \varepsilon_{G_2^*} u_2' :: G_2^* : G_1^* \leqslant G_2^*$. By the rule ($\leqslant\text{asc}_\varepsilon$), we are required to prove that $\varepsilon_{G_1^*} \leqslant \varepsilon_{G_2^*}$, $\Omega \vdash u_1' \leqslant u_2' : G_1^* \leqslant G_2^*$ and $G_1^* \sqsubseteq G_2^*$. Since $G_1^* \leqslant G_2^*$ and Proposition 10.22, we know that $\varepsilon_{G_1^*} \leqslant \varepsilon_{G_2^*}$. Also, by Proposition 10.23 and $G_1^* \leqslant G_2^*$, we now that $G_1^* \sqsubseteq G_2^*$. Therefore, we only have required to prove that $\Omega \vdash u_1' \leqslant u_2' : G_1^* \leqslant G_2^*$. We follow by case analysis on $\Omega \vdash u_1 : G_1^* \leqslant_v u_2 : G_2^*$. We only take into account the package, where $u_i = \text{packu}\langle G_i', v_i\rangle \text{ as } \exists X.G_i''$ and $G_i^* = \exists X.G_i''$, where $\exists X.G_1'' \leqslant \exists X.G_2''$. We know that

$$(\leqslant\text{pack})\frac{G_1' \leqslant G_2' \qquad \Omega \vdash \triangleright v_1 : G_1 \leqslant \Xi_2 \triangleright v_2 : G_2 \qquad \exists X.G_1'' \sqsubseteq \exists X.G_2'' \qquad G_1 \sqcap G_1''[G_1'/X] \leqslant G_2 \sqcap G_2''[G_2'/X]}{\Omega \vdash \triangleright\text{pack}\langle G_1', v_1\rangle \text{ as } \exists X.G_1'' : \exists X.G_1'' \leqslant \Xi_2 \triangleright \text{pack}\langle G_2', v_2\rangle \text{ as } \exists X.G_2'' : \exists X.G_2''}$$

$$(\text{Gpack})\frac{\Delta;\Gamma_1 \vdash v_1 :: G_1''[G_1'/X] \rightsquigarrow v_1'' : G_1''[G_1'/X]}{\Delta;\Gamma_1 \vdash \text{pack}\langle G_1', v_1\rangle \text{ as } \exists X.G_1'' \rightsquigarrow \text{packu}\langle G_1', v_1''\rangle \text{ as } \exists X.G_1'' : \exists X.G_1''}$$

$$(\text{Gpack})\frac{\Delta;\Gamma_2 \vdash v_2 :: G_2''[G_2'/X] \rightsquigarrow v_2'' : G_2''[G_2'/X]}{\Delta;\Gamma_2 \vdash \text{pack}\langle G_2', v_2\rangle \text{ as } \exists X.G_2'' \rightsquigarrow \text{packu}\langle G_2', v_2''\rangle \text{ as } \exists X.G_2'' : \exists X.G_2''}$$

We have to prove that $\Omega \vdash \triangleright\text{packu}\langle G_1', v_1''\rangle \text{ as } \exists X.G_1'' : \exists X.G_1'' \leqslant \Xi_2 \triangleright \text{packu}\langle G_2', v_2''\rangle \text{ as } \exists X.G_2'' : \exists X.G_2''$, or what is the same by the rule ($\leqslant\text{packu}_\varepsilon$), we have to prove that $G_1' \leqslant G_2'$, $\Omega \vdash \triangleright v_1'' : G_1''[G_1'/X] \leqslant \Xi_2 \triangleright v_2'' : G_2''[G_2'/X]$ and $\exists X.G_1'' \sqsubseteq \exists X.G_2''$. By premise, $G_1' \leqslant G_2'$ and $\exists X.G_1'' \sqsubseteq \exists X.G_2''$ (Proposition 10.16) follows immediately. Therefore, we only have required to prove that $\Omega \vdash \triangleright v_1'' : G_1''[G_1'/X] \leqslant \Xi_2 \triangleright v_2'' : G_2''[G_2'/X]$, which follows by the induction hypothesis. We know that

$$v_1'' = \varepsilon_1 v_1' :: G_1''[G_1'/X] \text{ where } \varepsilon_1 = \mathcal{I}(G_1, G_1''[G_1'/X])$$

$$v_2'' = \varepsilon_2 v_2' :: G_2''[G_2'/X] \text{ where } \varepsilon_2 = \mathcal{I}(G_2, G_2''[G_2'/X])$$

where $\Delta;\Gamma_i \vdash v_i \rightsquigarrow_v v_i' : G_i$, and therefore $\Omega \vdash v_1' \leqslant v_2' : G_1 \leqslant G_2$.

By rule ($\leqslant\text{asc}_\varepsilon$), we are required to prove that $\varepsilon_1 \leqslant \varepsilon_2$, $\Omega \vdash v_1' \leqslant v_2' : G_1 \leqslant G_2$ and $G_1''[G_1'/X] \sqsubseteq G_2''[G_2'/X]$. By induction hypothesis on $\Omega \vdash \Xi_1 \triangleright v_1 : G_1 \leqslant \Xi_2 \triangleright v_2 : G_2$, we know that $\Omega \vdash v_1' \leqslant v_2' : G_1 \leqslant G_2$. By Proposition 10.26, $G_1'' \leqslant G_2''$ and $G_1' \leqslant G_2'$, we know that $G_1''[G_1'/X] \leqslant G_2''[G_2'/X]$, and therefore $G_1''[G_1'/X] \sqsubseteq G_2''[G_2'/X]$. By Proposition 10.14 and $G_1 \sqcap G_1''[G_1'/X] \leqslant G_2 \sqcap G_2''[G_2'/X]$, we know that

$$\varepsilon_1 = \mathcal{I}(G_1, G_1''[G_1'/X]) = \mathcal{I}(G_1 \sqcap G_1''[G_1'/X], G_1 \sqcap G_1''[G_1'/X]) \leqslant$$
$$\mathcal{I}(G_2 \sqcap G_2''[G_2'/X], G_2 \sqcap G_2''[G_2'/X]) = \mathcal{I}(G_2, G_2''[G_2'/X]) = \varepsilon_2$$

Therefore, the results holds.

Case (⩽ascv). We know that

$$(\leqslant\text{ascv})\frac{\Omega \vdash u_1 : G_1^{**} \leqslant_v u_2 : G_2^{**} \qquad G_1^{**} \sqcap G_1^* \leqslant G_2^{**} \sqcap G_2^* \qquad G_1^* \sqsubseteq G_2^*}{\Omega \vdash u_1 :: G_1^* : G_1^* \leqslant u_2 :: G_2^* : G_2^*}$$

$$(\text{Gascu})\frac{\Delta;\Gamma_1 \vdash u_1 \rightsquigarrow u_1' : G_1^{**} \qquad \varepsilon_1 = \mathcal{I}(G_1^{**}, G_1^*)}{\Delta;\Gamma_1 \vdash u_1 :: G_1^* \rightsquigarrow \varepsilon_1 u_1' :: G_1^* : G_1^*}$$

$$\text{(Gascu)}\frac{\Delta;\Gamma_2 \vdash u_2 \rightsquigarrow u_2' : G_2^{**} \qquad \varepsilon_2 = \mathcal{I}(G_2^{**}, G_2^*)}{\Delta;\Gamma_2 \vdash u_2 :: G_2^* \rightsquigarrow \varepsilon_2 u_2' :: G_2^* : G_2^*}$$

We have to prove that $\Omega \vdash \varepsilon_1 u_1' :: G_1^* \leqslant \varepsilon_2 u_2' :: G_2^* : G_1^* \leqslant G_2^*$, or what is the same by the rule ($\leqslant$asc$_\varepsilon$), we have to prove that $\varepsilon_1 \leqslant \varepsilon_2$, $\Omega \vdash u_1' \leqslant u_2' : G_1^{**} \leqslant G_2^{**}$ and $G_1^* \sqsubseteq G_2^*$. By Proposition 10.13, we know that $\varepsilon_1 = \mathcal{I}(G_1^{**}, G_1^*) = \mathcal{I}(G_1^{**} \sqcap G_1^*, G_1^{**} \sqcap G_1^*)$ and $\varepsilon_2 = \mathcal{I}(G_2^{**}, G_2^*) = \mathcal{I}(G_2^{**} \sqcap G_2^*, G_2^{**} \sqcap G_2^*)$. Since $G_1^{**} \sqcap G_1^* \leqslant G_2^{**} \sqcap G_2^*$, then $\varepsilon_1 = \mathcal{I}(G_1^{**}, G_1^*) = \mathcal{I}(G_1^{**} \sqcap G_1^*, G_1^{**} \sqcap G_1^*) \leqslant \mathcal{I}(G_2^{**} \sqcap G_2^*, G_2^{**} \sqcap G_2^*) = \mathcal{I}(G_2^{**}, G_2^*) = \varepsilon_2$, by Proposition 10.14. Thus, we only have to prove that $\Omega \vdash u_1' \leqslant u_2' : G_1^{**} \leqslant G_2^{**}$, and we know that $\Omega \vdash u_1' : G_1^{**} \leqslant_v u_2' : G_2^{**}$. We follow by case analysis on $\Omega \vdash u_1 : G_1^{**} \leqslant_v u_2 : G_2^{**}$. We only take into account the package, where $u_i = \mathsf{pack}\langle G_i', v_i\rangle$ as $\exists X.G_i''$ and $G_i^* = \exists X.G_i''$, where $\exists X.G_1'' \leqslant \exists X.G_2''$. We know that

$$\text{($\leqslant$pack)}\frac{G_1' \leqslant G_2' \qquad \Omega \vdash \triangleright v_1 : G_1 \leqslant \Xi_2 \triangleright v_2 : G_2 \qquad \exists X.G_1'' \sqsubseteq \exists X.G_2'' \qquad G_1 \sqcap G_1''[G_1'/X] \leqslant G_2 \sqcap G_2''[G_2'/X]}{\Omega \vdash \triangleright \mathsf{pack}\langle G_1', v_1\rangle \text{ as } \exists X.G_1'' : \exists X.G_1'' \leqslant \Xi_2 \triangleright \mathsf{pack}\langle G_2', v_2\rangle \text{ as } \exists X.G_2'' : \exists X.G_2''}$$

$$\text{(Gpack)}\frac{\Delta;\Gamma_1 \vdash v_1 :: G_1''[G_1'/X] \rightsquigarrow v_1'' : G_1''[G_1'/X]}{\Delta;\Gamma_1 \vdash \mathsf{pack}\langle G_1', v_1\rangle \text{ as } \exists X.G_1'' \rightsquigarrow \mathsf{packu}\langle G_1', v_1''\rangle \text{ as } \exists X.G_1'' : \exists X.G_1''}$$

$$\text{(Gpack)}\frac{\Delta;\Gamma_2 \vdash v_2 :: G_2''[G_2'/X] \rightsquigarrow v_2'' : G_2''[G_2'/X]}{\Delta;\Gamma_2 \vdash \mathsf{pack}\langle G_2', v_2\rangle \text{ as } \exists X.G_2'' \rightsquigarrow \mathsf{packu}\langle G_2', v_2''\rangle \text{ as } \exists X.G_2'' : \exists X.G_2''}$$

We have to prove that $\Omega \vdash \triangleright \mathsf{packu}\langle G_1', v_1''\rangle$ as $\exists X.G_1'' : \exists X.G_1'' \leqslant \Xi_2 \triangleright \mathsf{packu}\langle G_2', v_2''\rangle$ as $\exists X.G_2'' : \exists X.G_2''$, or what is the same by the rule ($\leqslant$packu$_\varepsilon$), we have to prove that $G_1' \leqslant G_2'$, $\Omega \vdash \triangleright v_1'' : G_1''[G_1'/X] \leqslant \Xi_2 \triangleright v_2'' : G_2''[G_2'/X]$ and $\exists X.G_1'' \sqsubseteq \exists X.G_2''$. By premise, $G_1' \leqslant G_2'$ and $\exists X.G_1'' \sqsubseteq \exists X.G_2''$ (Proposition 10.16) follows immediately. Therefore, we only have required to prove that $\Omega \vdash \triangleright v_1'' : G_1''[G_1'/X] \leqslant \Xi_2 \triangleright v_2'' : G_2''[G_2'/X]$, which follows by the induction hypothesis.

We know that

$$v_1'' = \varepsilon_1 v_1' :: G_1''[G_1'/X] \text{ where } \varepsilon_1 = \mathcal{I}(G_1, G_1''[G_1'/X])$$

$$v_2'' = \varepsilon_2 v_2' :: G_2''[G_2'/X] \text{ where } \varepsilon_2 = \mathcal{I}(G_2, G_2''[G_2'/X])$$

where $\Delta;\Gamma_i \vdash v_i \rightsquigarrow_v v_i' : G_i$, and therefore $\Omega \vdash v_1' \leqslant v_2' : G_1 \leqslant G_2$.

By rule ($\leqslant$asc$_\varepsilon$), we are required to prove that $\varepsilon_1 \leqslant \varepsilon_2$, $\Omega \vdash v_1' \leqslant v_2' : G_1 \leqslant G_2$ and $G_1''[G_1'/X] \sqsubseteq G_2''[G_2'/X]$. By induction hypothesis on $\Omega \vdash \Xi_1 \triangleright v_1 : G_1 \leqslant \Xi_2 \triangleright v_2 : G_2$, we know that $\Omega \vdash v_1' \leqslant v_2' : G_1 \leqslant G_2$. By Proposition 10.26, $G_1'' \leqslant G_2''$ and $G_1' \leqslant G_2'$, we know that $G_1''[G_1'/X] \leqslant G_2''[G_2'/X]$, and therefore $G_1''[G_1'/X] \sqsubseteq G_2''[G_2'/X]$. By Proposition 10.14 and $G_1 \sqcap G_1''[G_1'/X] \leqslant G_2 \sqcap G_2''[G_2'/X]$, we know that

$$\varepsilon_1 = \mathcal{I}(G_1, G_1''[G_1'/X]) = \mathcal{I}(G_1 \sqcap G_1''[G_1'/X], G_1 \sqcap G_1''[G_1'/X]) \leqslant$$
$$\mathcal{I}(G_2 \sqcap G_2''[G_2'/X], G_2 \sqcap G_2''[G_2'/X]) = \mathcal{I}(G_2, G_2''[G_2'/X]) = \varepsilon_2$$

Therefore, the results holds.

Case ($\leqslant$pack). We know that

$$\text{($\leqslant$pack)}\frac{G_1' \leqslant G_2' \qquad \Omega \vdash \triangleright t_1 : G_1 \leqslant \Xi_2 \triangleright t_2 : G_2 \qquad \exists X.G_1'' \leqslant \exists X.G_2'' \qquad G_1 \sqcap G_1''[G_1'/X] \leqslant G_2 \sqcap G_2''[G_2'/X]}{\Omega \vdash \triangleright \mathsf{pack}\langle G_1', t_1\rangle \text{ as } \exists X.G_1'' : \exists X.G_1'' \leqslant \Xi_2 \triangleright \mathsf{pack}\langle G_2', t_2\rangle \text{ as } \exists X.G_2'' : \exists X.G_2''}$$

$$\text{(Gpack)}\frac{\Delta;\Gamma_1 \vdash t_1 \rightsquigarrow t_1' : G_1 \qquad t_1'' = norm(t_1', G_1, G_1''[G_1'/X])}{\Delta;\Gamma_1 \vdash \mathsf{pack}\langle G_1', t_1\rangle \text{ as } \exists X.G_1'' \rightsquigarrow \mathsf{pack}\langle G_1', t_1''\rangle \text{ as } \exists X.G_1'' : \exists X.G_1''}$$

$$\text{(Gpack)}\frac{\Delta;\Gamma_2 \vdash t_2 \rightsquigarrow t_2' : G_2 \qquad t_2'' = norm(t_2', G_2, G_2''[G_2'/X])}{\Delta;\Gamma_2 \vdash \mathsf{pack}\langle G_2', t_2\rangle \text{ as } \exists X.G_2'' \rightsquigarrow \mathsf{pack}\langle G_2', t_2''\rangle \text{ as } \exists X.G_2'' : \exists X.G_2''}$$

We have to prove that $\Omega \vdash \triangleright\mathsf{pack}\langle G_1', t_1''\rangle \text{ as } \exists X.G_1'' : \exists X.G_1'' \leqslant \Xi_2 \triangleright \mathsf{pack}\langle G_2', t_2''\rangle \text{ as } \exists X.G_2'' : \exists X.G_2''$, or what is the same by the rule ($\leqslant$pack$_\varepsilon$), we have to prove that $G_1' \leqslant G_2'$, $\Omega \vdash \triangleright t_1'' : G_1''[G_1'/X] \leqslant \Xi_2 \triangleright t_2'' : G_2''[G_2'/X]$ and $\exists X.G_1'' \leqslant \exists X.G_2''$. By premise, $G_1' \leqslant G_2'$ and $\exists X.G_1'' \leqslant \exists X.G_2''$ (Proposition 10.16) follows immediately. Therefore, we only have required to prove that $\Omega \vdash \triangleright t_1'' : G_1''[G_1'/X] \leqslant \Xi_2 \triangleright t_2'' : G_2''[G_2'/X]$. We know that

$$t_1'' = norm(t_1', G_1, G_1''[G_1'/X]) = \varepsilon_1 t_1' :: G_1''[G_1'/X] \text{ where } \varepsilon_1 = \mathcal{I}(G_1, G_1''[G_1'/X])$$

$$t_2'' = norm(t_2', G_2, G_2''[G_2'/X]) = \varepsilon_2 t_2' :: G_2''[G_2'/X] \text{ where } \varepsilon_2 = \mathcal{I}(G_2, G_2''[G_2'/X])$$

By rule ($\leqslant$asc$_\varepsilon$), we are required to prove that $\varepsilon_1 \leqslant \varepsilon_2$, $\Omega \vdash t_1' \leqslant t_2' : G_1 \leqslant G_2$ and $G_1''[G_1'/X] \sqsubseteq G_2''[G_2'/X]$. By induction hypothesis on $\Omega \vdash \triangleright t_1 : G_1 \leqslant \Xi_2 \triangleright t_2 : G_2$, we know that $\Omega \vdash t_1' \leqslant t_2' : G_1 \leqslant G_2$. By Proposition 10.26, $G_1'' \leqslant G_2''$ and $G_1' \leqslant G_2'$, we know that $G_1''[G_1'/X] \leqslant G_2''[G_2'/X]$, and therefore $G_1''[G_1'/X] \sqsubseteq G_2''[G_2'/X]$. By Proposition 10.14 and $G_1 \sqcap G_1''[G_1'/X] \leqslant G_2 \sqcap G_2''[G_2'/X]$, we know that

$$\begin{aligned}\varepsilon_1 = \mathcal{I}(G_1, G_1''[G_1'/X]) = \mathcal{I}(G_1 \sqcap G_1''[G_1'/X], G_1 \sqcap G_1''[G_1'/X]) \leqslant \\ \mathcal{I}(G_2 \sqcap G_2''[G_2'/X], G_2 \sqcap G_2''[G_2'/X]) = \mathcal{I}(G_2, G_2''[G_2'/X]) = \varepsilon_2\end{aligned}$$

Therefore, the results holds.

Case (unpack). We know that

$$\text{($\leqslant$unpack)}\frac{\Omega \vdash \triangleright t_{11} : G_1 \leqslant \Xi_2 \triangleright t_{21} : G_2 \qquad \Omega, x : schm_e^\sharp(G_1) \sqsubseteq schm_e^\sharp(G_2) \vdash \triangleright t_{12} : G_1' \leqslant \Xi_2 \triangleright t_{22} : G_2'}{\Omega \vdash \triangleright \mathsf{unpack}\langle X, x\rangle = t_{11} \text{ in } t_{12} : G_1' \leqslant \Xi_2 \triangleright \mathsf{unpack}\langle X, x\rangle = t_{21} \text{ in } t_{22} : G_2'}$$

$$\text{(Gunpack)}\frac{\begin{array}{c}\Delta;\Gamma_1 \vdash t_{11} \rightsquigarrow t_{11}' : G_1 \qquad t_{11}'' = norm(t_{11}', G_1, \exists var^\sharp(G_1).schm_e^\sharp(G_1)) \\ \Delta;\Gamma_1, x : schm_e^\sharp(G_1) \vdash t_{12} \rightsquigarrow t_{12}' : G_1'\end{array}}{\Delta;\Gamma_1 \vdash \mathsf{unpack}\langle X, x\rangle = t_{11} \text{ in } t_{12} \rightsquigarrow \mathsf{unpack}\langle X, x\rangle = t_{11}'' \text{ in } t_{12}' : G_1'}$$

$$\text{(Gunpack)}\frac{\begin{array}{c}\Delta;\Gamma_2 \vdash t_{21} \rightsquigarrow t_{21}' : G_2 \qquad t_{21}'' = norm(t_{21}', G_2, \exists var^\sharp(G_2).schm_e^\sharp(G_2)) \\ \Delta;\Gamma_2, x : schm_e^\sharp(G_2) \vdash t_{22} \rightsquigarrow t_{22}' : G_2'\end{array}}{\Delta;\Gamma_2 \vdash \mathsf{unpack}\langle X, x\rangle = t_{21} \text{ in } t_{22} \rightsquigarrow \mathsf{unpack}\langle X, x\rangle = t_{21}'' \text{ in } t_{22}' : G_2'}$$

We have to prove that $\Omega \vdash \mathsf{unpack}\langle X, x\rangle = t_{11}'' \text{ in } t_{12}' \leqslant \mathsf{unpack}\langle X, x\rangle = t_{21}'' \text{ in } t_{22}' : G_1' \leqslant G_2'$, or what is the same by the rule ($\leqslant$unpack$_\varepsilon$), we have to prove that $\Omega \vdash t_{11}'' \leqslant t_{21}'' : \exists var^\sharp(G_1).schm_e^\sharp(G_1) \leqslant \exists var^\sharp(G_2).schm_e^\sharp(G_2)$ and $\Omega, x : schm_e^\sharp(G_1) \sqsubseteq schm_e^\sharp(G_2) \vdash t_{12}' \leqslant t_{22}' : G_1' \leqslant G_2'$. By the induction hypothesis on $\Omega, x : schm_e^\sharp(G_1) \sqsubseteq schm_e^\sharp(G_2) \vdash \triangleright t_{12} : G_1' \leqslant \Xi_2 \triangleright t_{22} : G_2'$, we know that $\Omega, x : schm_e^\sharp(G_1) \sqsubseteq schm_e^\sharp(G_2) \vdash t_{12}' \leqslant t_{22}' : G_1' \leqslant G_2'$. Therefore, we only are required to prove that $\Omega \vdash t_{11}'' \leqslant t_{21}'' : \exists var^\sharp(G_1).schm_e^\sharp(G_1) \leqslant \exists var^\sharp(G_2).schm_e^\sharp(G_2)$. We know that

$$t_{11}'' = norm(t_{11}', G_1, \exists var^\sharp(G_1).schm_e^\sharp(G_1)) = \varepsilon_1 t_{11}' :: \exists var^\sharp(G_1).schm_e^\sharp(G_1)$$

$$\text{where } \varepsilon_1 = \mathcal{I}(G_1, \exists var^\sharp(G_1).schm_e^\sharp(G_1)) = \mathcal{I}(\exists var^\sharp(G_1).schm_e^\sharp(G_1), \exists var^\sharp(G_1).schm_e^\sharp(G_1)) =$$

$$\varepsilon_{\exists var^\sharp(G_1).schm_e^\sharp(G_1)}$$

$$t_{21}'' = norm(t_{21}', G_2, \exists var^\sharp(G_2).schm_e^\sharp(G_2)) = \varepsilon_2 t_{21}' :: \exists var^\sharp(G_2).schm_e^\sharp(G_2)$$

$$\text{where } \varepsilon_2 = \mathcal{I}(G_2, \exists var^\sharp(G_2).schm_e^\sharp(G_2)) = \mathcal{I}(\exists var^\sharp(G_2).schm_e^\sharp(G_2), \exists var^\sharp(G_2).schm_e^\sharp(G_2)) =$$

$$\varepsilon_{\exists var^\sharp(G_2).schm_e^\sharp(G_2)}$$

By induction hypothesis on $\Omega \vdash t_{11} : G_1 \leqslant t_{21} : G_2$, we know that $\Omega \vdash \triangleright t'_{11} : G_1 \leqslant \Xi_2 \triangleright t'_{21} : G_2$, and by Proposition 10.16, we know that $G_1 \sqsubseteq G_2$, thus $\exists var^\sharp(G_1).schm^\sharp_e(G_1) \sqsubseteq \exists var^\sharp(G_2).schm^\sharp_e(G_2)$. Therefore, we only have to prove by rule ($\leqslant$Masc$_\varepsilon$) that $\varepsilon_1 \sqsubseteq \varepsilon_2$. But, by Proposition 10.15 and $\exists var^\sharp(G_1).schm^\sharp_e(G_1) \sqsubseteq \exists var^\sharp(G_2).schm^\sharp_e(G_2)$ the results holds.

$\square$

Proposition 10.13. $\mathcal{I}_\Xi(G_1 \sqcap G_2, G_1 \sqcap G_2) = \mathcal{I}_\Xi(G_1, G_2)$

Proof. By the definition of $\sqcap$ and $\mathcal{I}_\Xi(G_1, G_2)$. $\square$

Proposition 10.14. *If* $G_1 \sqcap G_2 \leqslant G'_1 \sqcap G'_2$, *then*

$$\mathcal{I}_\Xi(G_1, G_2) = \mathcal{I}_\Xi(G_1 \sqcap G_2, G_1 \sqcap G_2) \leqslant \mathcal{I}_\Xi(G'_1 \sqcap G'_2, G'_1 \sqcap G'_2) = \mathcal{I}_\Xi(G'_1, G'_2)$$

Proof. By Proposition 10.13 and the definition of $\leqslant$ in evidence. $\square$

Proposition 10.15. *If* $G_1 \leqslant G_2$, *then*

$$\mathcal{I}_\Xi(G_1, G_1) \sqsubseteq \mathcal{I}_\Xi(G_2, G_2)$$

Proof. By the definition of $\mathcal{I}_\Xi$ and the $\sqsubseteq$ in evidence. $\square$

Proposition 10.16. $\Omega \vdash \Xi_1 \triangleright s_1 : G_1 \leqslant \Xi_2 \triangleright s_2 : G_2$ *then* $G_1 \sqsubseteq G_2$.

Proof. By the definition of $\sqcap$ and $\mathcal{I}_\Xi(G_1, G_2)$. $\square$

Proposition 10.17. *If* $\Xi_1 \vdash t^*_1 \leqslant \Xi_2 \vdash t^*_2$ *and* $\Xi_1 \triangleright t^*_1 \longrightarrow \Xi'_1 \triangleright t^{**}_1$, *then* $\Xi_2 \triangleright t^*_2 \longrightarrow \Xi'_2 \triangleright t^{**}_2$ *and* $\Xi'_1 \vdash t^{**}_1 \leqslant \Xi'_2 \vdash t^{**}_2$.

Proof. If $\Xi_1 \vdash t^*_1 \leqslant \Xi_2 \vdash t^*_2$, we know that $\vdash t^*_1 \leqslant t^*_2 : G^*_1 \leqslant G^*_2$, $\Xi_1 \leqslant \Xi_2$, $\Xi_1 \vdash t^*_1 : G^*_1$ and $\Xi_2 \vdash t^*_2 : G^*_2$. We follow by induction on $\vdash t^*_1 \leqslant t^*_2 : G^*_1 \leqslant G^*_2$. We avoid the notation $\vdash t_1 \leqslant t_2 : G_1 \leqslant G_2$, and use $t_1 \leqslant t_2$ instead, for simplicity, when the typing environments are not relevant. We only take into account the existential unpack case.

Case (pack). We know that

$$(\leqslant\text{pack})\,\frac{G''_1 \leqslant G''_2 \qquad \vdash \Xi_1 \triangleright t_{11} : G_{11}[G''_1/X] \leqslant \Xi_2 \triangleright t_{22} : G_{22}[G''_2/X] \qquad \exists X.G_{11} \leqslant \exists X.G_{22}}{\vdash \Xi_1 \triangleright \mathsf{pack}\langle G''_1, t_{11}\rangle \text{ as } \exists X.G_{11} : \exists X.G_{11} \leqslant \Xi_2 \triangleright \mathsf{pack}\langle G''_2, t_{22}\rangle \text{ as } \exists X.G_{22} : \exists X.G_{22}}$$

Also, since $\Xi_1 \triangleright t^*_1 \longrightarrow \Xi'_1 \triangleright t^*_1$, we know that $t_{11} = v_{11}$. By Proposition 10.27 and $\vdash \Xi_1 \triangleright t_{11} : G_{11}[G''_1/X] \leqslant \Xi_2 \triangleright t_{22} : G_{22}[G''_2/X]$, we know that $t_{22} = v_{22}$.

By the reduction rules, we know that

$$\Xi_1 \triangleright \mathsf{pack}\langle G''_1, v_{11}\rangle \text{ as } \exists X.G_{11} \longrightarrow \Xi_1 \triangleright \varepsilon_{\exists X.G_{11}}\mathsf{packu}\langle G''_1, v_{11}\rangle \text{ as } \exists X.G_{11} :: \exists X.G_{11}$$

$$\Xi_2 \triangleright \mathsf{pack}\langle G''_2, v_{22}\rangle \text{ as } \exists X.G_{22} \longrightarrow \Xi_2 \triangleright \varepsilon_{\exists X.G_{22}}\mathsf{packu}\langle G''_2, v_{22}\rangle \text{ as } \exists X.G_{22} :: \exists X.G_{22}$$

We are required to prove that

$$\vdash \varepsilon_{\exists X.G_{11}}\mathsf{packu}\langle G''_1, v_{11}\rangle \text{ as } \exists X.G_{11} :: \exists X.G_{11} \leqslant:\leqslant$$
$$\varepsilon_{\exists X.G_{22}}\mathsf{packu}\langle G''_2, v_{22}\rangle \text{ as } \exists X.G_{22} :: \exists X.G_{22} : \exists X.G_{11} \leqslant \exists X.G_{22}$$

This follows immediately by rules ($\leqslant$packu$_\varepsilon$) and ($\leqslant$asc$_\varepsilon$). Note that $\varepsilon_{\exists X.G_{11}} \leqslant \varepsilon_{\exists X.G_{22}}$, by Lemma 10.15.

Case (unpack). We know that

$$(\leqslant\text{unpack})\frac{\vdash \Xi_1 \triangleright t_{11} : \exists X.G_1 \leqslant \Xi_2 \triangleright t_{21} : \exists X.G_2 \qquad x : G_1 \sqsubseteq G_2 \vdash \Xi_1 \triangleright t_{12} : G_1' \leqslant \Xi_2 \triangleright t_{22} : G_2'}{\vdash \Xi_1 \triangleright \mathsf{unpack}\langle X, x\rangle = t_{11} \ \mathsf{in}\ t_{12} : G_1' \leqslant \Xi_2 \triangleright \mathsf{unpack}\langle X, x\rangle = t_{21} \ \mathsf{in}\ t_{22} : G_2'}$$

Also, since $\Xi_1 \triangleright t_1^* \longrightarrow \Xi_1' \triangleright t_1^*$, we know that $t_{11} = \varepsilon_{11}\mathsf{packu}\langle G_1'', \varepsilon_1 u_1 :: G_{11}[G_1''/X]\rangle$ as $\exists X.G_{11} :: \exists X.G_1$. By Proposition 10.27 and $\vdash \Xi_1 \triangleright t_{11} : \exists X.G_1 \leqslant \Xi_2 \triangleright t_{21} : \exists X.G_2$, we know that $t_{21} = \varepsilon_{22}\mathsf{packu}\langle G_2'', \varepsilon_2 u_2 :: G_{22}[G_2''/X]\rangle$ as $\exists X.G_{22} :: \exists X.G_2$. By the reduction rules, we know that

$$\Xi_1 \triangleright \mathsf{unpack}\langle X, x\rangle = t_{11} \ \mathsf{in}\ t_{12} \longrightarrow \Xi_1' \triangleright t_{12}[\hat{\alpha}/X][((\varepsilon_1 \circ\!\!\!\circ\, \varepsilon_{11}[\hat{G_1''}, \hat{\alpha}])u_1 :: G_1[\alpha/X])/x]$$

where $\Xi_1' = \Xi_1, \alpha := G_1''$ and $\hat{\alpha_1} = \mathit{lift}_{\Xi_1'}(\alpha)$.

We know that $\varepsilon_{11} \leqslant \varepsilon_{22}$, $\Xi_1' \leqslant \Xi_2'$ and $G_1'' \leqslant G_2''$, therefore by Proposition 10.19, we know that $\varepsilon_{11}[\hat{G_1''}, \hat{\alpha}] \leqslant \varepsilon_{22}[\hat{G_2''}, \hat{\alpha}]$. Therefore, we know that $(\varepsilon_1 \circ\!\!\!\circ\, \varepsilon_{11}[\hat{G_1''}, \hat{\alpha}]) \leqslant (\varepsilon_2 \circ\!\!\!\circ\, \varepsilon_{22}[\hat{G_2''}, \hat{\alpha}])$, by Proposition 10.20 and $\varepsilon_1 \leqslant \varepsilon_2$.

Therefore, we know that

$$\Xi_2 \triangleright \mathsf{unpack}\langle X, x\rangle = t_{21} \ \mathsf{in}\ t_{22} \longrightarrow \Xi_2' \triangleright t_{22}[\hat{\alpha}/X][((\varepsilon_2 \circ\!\!\!\circ\, \varepsilon_{22}[\hat{G_2''}, \hat{\alpha}])u_2 :: G_2[\alpha/X])/x]$$

where $\Xi_2' = \Xi_2, \alpha := G_2''$ and $\hat{\alpha_2} = \mathit{lift}_{\Xi_2'}(\alpha)$.

Since $\Xi_1 \leqslant \Xi_2$ and $G_1'' \leqslant G_2''$, we know that $\Xi_1' \leqslant \Xi_2'$. Therefore, we only are required to prove that

$$t_{12}[\hat{\alpha}/X][((\varepsilon_1 \circ\!\!\!\circ\, \varepsilon_{11}[\hat{G_1''}, \hat{\alpha}])u_1 :: G_1[\alpha/X])/x] : G_1' \leqslant t_{22}[\hat{\alpha}/X][((\varepsilon_2 \circ\!\!\!\circ\, \varepsilon_{22}[\hat{G_2''}, \hat{\alpha}])u_2 :: G_2[\alpha/X])/x] : G_2'$$

By Proposition 10.21 we know that $t_{12}[\hat{\alpha_1}/X] \leqslant t_{22}[\hat{\alpha_2}/X]$.

We know that $((\varepsilon_1 \circ\!\!\!\circ\, \varepsilon_{11}[\hat{G_1''}, \hat{\alpha}])u_1 :: G_1[\alpha/X]) \leqslant ((\varepsilon_2 \circ\!\!\!\circ\, \varepsilon_{22}[\hat{G_2''}, \hat{\alpha}])u_2 :: G_2[\alpha/X])$, by the Rule ($\leqslant$asc$_\varepsilon$) and since $u_1 \leqslant u_2$, $(\varepsilon_1 \circ\!\!\!\circ\, \varepsilon_{11}[\hat{G_1''}, \hat{\alpha}]) \leqslant (\varepsilon_2 \circ\!\!\!\circ\, \varepsilon_{22}[\hat{G_2''}, \hat{\alpha}])$ and $G_1[\alpha/X] \sqsubseteq G_2[\alpha/X]$ (by Proposition 10.24 and Proposition 10.25). Finally, by Proposition 10.18 the result holds.

□

Proposition 10.18 (Substitution Preserves Precision). *If* $\Omega', x : G_1 \sqsubseteq G_2 \vdash s_1 \leqslant s_2 : G_1' \leqslant G_2'$ *and* $\Omega' \vdash v_1 \leqslant v_2 : G_1 \leqslant G_2$*, then* $\Omega' \vdash s_1[v_1/x] \leqslant s_2[v_2/x] : G_1' \leqslant G_2'$.

Proof. We follow by induction on $\Omega', x : G_1 \sqsubseteq G_2 \vdash t_1 \leqslant t_2 : G_1' \leqslant G_2'$. We avoid the notation $\Omega', x : G_1 \sqsubseteq G_2 \vdash t_1 \leqslant t_2 : G_1' \leqslant G_2'$, and use $t_1 \leqslant t_2$ instead, for simplicity, when the typing environments are not relevant. Let suppose that $\Omega = \Omega', x : G_1 \sqsubseteq G_2$.

Case (packu). We know that

$$(\leqslant\text{packu}_\varepsilon)\frac{G_1^{**} \leqslant G_2^{**} \qquad \Omega \vdash \Xi_1 \triangleright v_1' : G_1^*[G_1^{**}/X] \leqslant \Xi_2 \triangleright v_2' : G_2^*[G_2^{**}/X] \qquad \exists X.G_1^* \sqsubseteq \exists X.G_2^*}{\Omega \vdash \Xi_1 \triangleright \mathsf{packu}\langle G_1^{**}, v_1'\rangle \ \mathsf{as}\ \exists X.G_1^* : \exists X.G_1^* \leqslant \Xi_2 \triangleright \mathsf{packu}\langle G_2^{**}, v_2'\rangle \ \mathsf{as}\ \exists X.G_2^* : \exists X.G_2^*}$$

Note that we are required to prove that

$$\Omega \vdash \Xi_1 \triangleright \mathsf{packu}\langle G_1^{**}, v_1'[v_1/x]\rangle \ \mathsf{as}\ \exists X.G_1^* : \exists X.G_1^* \leqslant \Xi_2 \triangleright \mathsf{packu}\langle G_2^{**}, v_2'[v_2/x]\rangle \ \mathsf{as}\ \exists X.G_2^* : \exists X.G_2^*$$

or what is the same $\Omega \vdash \Xi_1 \triangleright v_1''[v_1/x] : G_1^*[G_1^{**}/X] \leqslant \Xi_2 \triangleright v_2''[v_2/x] : G_2^*[G_2^{**}/X]$. But the result follows immediately by the induction hypothesis on $\Omega \vdash \Xi_1 \triangleright v_1' : G_1^*[G_1^{**}/X] \leqslant \Xi_2 \triangleright v_2' : G_2^*[G_2^{**}/X]$.

Case (pack). We know that

$$(\leqslant\text{pack}_\varepsilon)\frac{G_1^{**} \leqslant G_2^{**} \qquad \Omega \vdash \Xi_1 \triangleright t_1 : G_1^*[G_1^{**}/X] \leqslant \Xi_2 \triangleright t_2 : G_2^*[G_2^{**}/X] \qquad \exists X.G_1^* \leqslant \exists X.G_2^*}{\Omega \vdash \Xi_1 \triangleright \mathsf{pack}\langle G_1^{**}, t_1\rangle \ \mathsf{as}\ \exists X.G_1^* : \exists X.G_1^* \leqslant \Xi_2 \triangleright \mathsf{pack}\langle G_2^{**}, t_2\rangle \ \mathsf{as}\ \exists X.G_2^* : \exists X.G_2^*}$$

Note that we are required to prove that

$$\Omega \vdash \Xi_1 \triangleright \mathsf{pack}\langle G_1^{**}, t_1[v_1/x]\rangle \text{ as } \exists X.G_1^* : \exists X.G_1^* \leqslant \Xi_2 \triangleright \mathsf{pack}\langle G_2^{**}, t_2[v_2/x]\rangle \text{ as } \exists X.G_2^* : \exists X.G_2^*$$

or what is the same $\Omega \vdash \Xi_1 \triangleright t_1[v_1/x] : G_1^*[G_1^{**}/X] \leqslant \Xi_2 \triangleright t_2[v_2/x] : G_2^*[G_2^{**}/X]$. But the result follows immediately by the induction hypothesis on $\Omega \vdash \Xi_1 \triangleright t_1 : G_1^*[G_1^{**}/X] \leqslant \Xi_2 \triangleright t_2 : G_2^*[G_2^{**}/X]$.

Case (unpack). We know that

$$(\leqslant\text{unpack}_\varepsilon)\frac{\Omega \vdash \Xi_1 \triangleright t_1 : \exists X.G_1^* \leqslant \Xi_2 \triangleright t_2 : \exists X.G_2^* \qquad \Omega, x : G_1^* \sqsubseteq G_2^* \vdash \Xi_1 \triangleright t_1' : G_1^{**} \leqslant \Xi_2 \triangleright t_2' : G_2^{**}}{\Omega \vdash \Xi_1 \triangleright \mathsf{unpack}\langle X, x\rangle = t_1 \text{ in } t_1' : G_1^{**} \leqslant \Xi_2 \triangleright \mathsf{unpack}\langle X, x\rangle = t_2 \text{ in } t_2' : G_2^{**}}$$

Note that we are required to prove that $\Omega' \vdash \Xi_1 \triangleright \mathsf{unpack}\langle X, x\rangle = t_1[v_1/x] \text{ in } t_1'[v_1/x] : G_1^{**} \leqslant \Xi_2 \triangleright \mathsf{unpack}\langle X, x\rangle = t_2[v_2/x] \text{ in } t_2'[v_2/x] : G_2^{**}$. Or what is the same $\Omega' \vdash \Xi_1 \triangleright t_1[v_1/x] : \exists X.G_1^* \leqslant \Xi_2 \triangleright t_2[v_2/x] : \exists X.G_1^*$ and $\Omega', x : G_1^* \sqsubseteq G_2^* \vdash \Xi_1 \triangleright t_1'[v_1/x] : G_1^{**} \leqslant \Xi_2 \triangleright t_2'[v_2/x] : G_2^{**}$. But the result follows immediately by the induction hypothesis on $\Omega \vdash \Xi_1 \triangleright t_1 : \exists X.G_1^* \leqslant \Xi_2 \triangleright t_2 : \exists X.G_2^*$ and $\Omega, x : G_1^* \sqsubseteq G_2^* \vdash \Xi_1 \triangleright t_1' : G_1^{**} \leqslant \Xi_2 \triangleright t_2' : G_2^{**}$.

□

Proposition 10.19. *If $\varepsilon_1 \leqslant \varepsilon_2$, $G_1 \leqslant G_2$, $\Xi_1 \leqslant \Xi_2$, $\alpha := G_1 \in \Xi_1$, $\alpha := G_2 \in \Xi_2$ and $\varepsilon_1[\hat{G}_1, \hat{\alpha}_1]$ is defined, then $\varepsilon_1[\hat{G}_1, \hat{\alpha}_1] \leqslant \varepsilon_2[\hat{G}_2, \hat{\alpha}_2]$, where $\hat{\alpha}_1 = lift_{\Xi_1}(\alpha)$, $\hat{\alpha}_2 = lift_{\Xi_2}(\alpha)$, $\hat{G}_1 = lift_{\Xi_1}(G_1)$ and $\hat{G}_2 = lift_{\Xi_2}(G_2)$.*

Proof. Note that $\hat{\alpha}_1 \leqslant \hat{\alpha}_2$ and $\hat{G}_1 \leqslant \hat{G}_2$ by Proposition 10.22. Suppose that $\varepsilon_1 = \langle \exists X.E, \exists X.E'\rangle$ and $\varepsilon_2 = \langle \exists X.E'', \exists X.E'''\rangle$ (since $\varepsilon_1[\hat{G}_1, \hat{\alpha}]$ is defined). We are required to prove that

$$\varepsilon_1[\hat{G}_1, \hat{\alpha}_1] = \langle E[\hat{G}_1/X], E'[\hat{\alpha}_1/X]\rangle \leqslant \langle E''[\hat{G}_2/X], E'''[\hat{\alpha}_2/X]\rangle = \varepsilon_2[\hat{G}_2, \hat{\alpha}_2]$$

Thus, we are required to prove that $E[\hat{G}_1/X] \leqslant E''[\hat{G}_2/X]$ and $E'[\hat{\alpha}_1/X] \leqslant E'''[\hat{\alpha}_2/X]$. Since $\varepsilon_1 \leqslant \varepsilon_2$, we know that $\langle \exists X.E, \exists X.E'\rangle \leqslant \langle \exists X.E'', \exists X.E'''\rangle$, and therefore $E \leqslant E''$ and $E' \leqslant E'''$. By Proposition 10.26 and $\hat{\alpha}_1 \leqslant \hat{\alpha}_2$ and $\hat{G}_1 \leqslant \hat{G}_2$, we know that $E[\hat{G}_1/X] \leqslant E''[\hat{G}_2/X]$ and $E'[\hat{\alpha}_1/X] \leqslant E'''[\hat{\alpha}_2/X]$. Therefore the result holds. □

Proposition 10.20 (Monotonicity of Evidence Transitivity). *If $\varepsilon_1 \leqslant \varepsilon_2$, $\varepsilon_3 \leqslant \varepsilon_4$, and $\varepsilon_1 \fatsemi \varepsilon_3$ is defined, then $\varepsilon_1 \fatsemi \varepsilon_3 \leqslant \varepsilon_2 \fatsemi \varepsilon_4$.*

Proof. By definition of consistent transitivity for = and the definition of precision. We only take into account the existential type case.

Case ([∃]- $\varepsilon_i = \langle \exists X.E_i, \exists X.E_i'\rangle$). By the definition of $\leqslant$, we know that $\langle E_1, E_1'\rangle \leqslant \langle E_2, E_2'\rangle$ and $\langle E_3, E_3'\rangle \leqslant \langle E_4, E_4'\rangle$. By the definition of transitivity we know that $\langle \exists X.E_1, \exists X.E_1'\rangle \fatsemi \langle \exists X.E_3, \exists X.E_3'\rangle = \langle \exists X.E_5, \exists X.E_5'\rangle$ and $\langle \exists X.E_2, \exists X.E_2'\rangle \fatsemi \langle \exists X.E_4, \exists X.E_4'\rangle = \langle \exists X.E_6, \exists X.E_6'\rangle$, where $\langle E_5, E_5'\rangle = \langle E_1, E_1'\rangle \fatsemi \langle E_3, E_3'\rangle$ and $\langle E_6, E_6'\rangle = \langle E_2, E_2'\rangle \fatsemi \langle E_4, E_4'\rangle$. Therefore, we are required to prove that $\langle E_5, E_5'\rangle \leqslant \langle E_6, E_6'\rangle$. But the result follows immediately by the induction hypothesis on $\langle E_1, E_1'\rangle \leqslant \langle E_2, E_2'\rangle$ and $\langle E_3, E_3'\rangle \leqslant \langle E_4, E_4'\rangle$.

□

Proposition 10.21 (Monotonicity of Evidence Substitution). *If $\Omega \vdash s_1^* \leqslant s_2^* : G_1^* \leqslant G_2^*$ and $\Xi_1 \leqslant \Xi_2$, then $\Omega[\alpha/X] \vdash s_1^*[\hat{\alpha}_1/X] \leqslant s_2^*[\hat{\alpha}_2/X] : G_1^*[\alpha/X] \leqslant G_2^*[\alpha/X]$, where $\alpha := G_1^{**} \in \Xi_1$, $\alpha := G_2^{**} \in \Xi_2$, $\hat{\alpha}_1 = lift_{\Xi_1}(\alpha)$ and $\hat{\alpha}_2 = lift_{\Xi_2}(\alpha)$.*

Proof. We follow by induction on $\Omega \vdash s_1^* \leqslant s_2^* : G_1^* \leqslant G_2^*$. We avoid the notation $\Omega \vdash s_1^* \leqslant s_2^* : G_1^*[\alpha/X] \leqslant G_2^*[\alpha/X]$, and use $s_1^* \leqslant s_2^*$ instead, for simplicity, when the typing environments are not relevant. We only take into account the cases related to existential types.

Case (packu). We know that

$$(\leqslant\text{packu}_\varepsilon)\dfrac{G_1' \leqslant G_2' \qquad \Omega \vdash \Xi_1 \triangleright v_1 : G_1[G_1'/Y] \leqslant \Xi_2 \triangleright v_2 : G_2[G_2'/Y] \qquad \exists Y.G_1 \sqsubseteq \exists Y.G_2}{\Omega \vdash \Xi_1 \triangleright \mathsf{packu}\langle G_1', v_1\rangle \text{ as } \exists Y.G_1 : \exists Y.G_1 \leqslant \Xi_2 \triangleright \mathsf{packu}\langle G_2', v_2\rangle \text{ as } \exists Y.G_2 : \exists Y.G_2}$$

We are required to show

$$\Omega[\alpha/X] \vdash \Xi_1 \triangleright \mathsf{packu}\langle G_1'[\alpha/X], v_1[\hat{\alpha_1}/X]\rangle \text{ as } \exists Y.G_1[\alpha/X] :\leqslant \Xi_2 \triangleright :$$

$$\mathsf{packu}\langle G_2', v_2[\hat{\alpha_1}/X]\rangle \text{ as } \exists Y.G_2 : \exists Y.G_1[\alpha/X] \leqslant \exists Y.G_2[\alpha/X]$$

Note that $G_1'[\alpha/X] \leqslant G_2'[\alpha/X]$ by Proposition 10.26 and $\exists Y.G_1[\alpha/X] \sqsubseteq \exists Y.G_2[\alpha/X]$ by Proposition 10.25. Therefore, we are required to prove $\Omega[\alpha/X] \vdash \Xi_1 \triangleright (v_1[\hat{\alpha_1}/X]) : G_1[G_1'/Y][\alpha/X] \leqslant \Xi_2 \triangleright (v_2[\hat{\alpha_2}/X]) : G_2[G_2'/Y][\alpha/X]$. But the results follows immediately by the induction hypothesis on $\Omega \vdash \Xi_1 \triangleright v_1 : G_1[G_1'/Y] \leqslant \Xi_2 \triangleright v_2 : G_2[G_2'/Y]$.

Case (pack). We know that

$$(\leqslant\text{pack}_\varepsilon)\dfrac{G_1' \leqslant G_2' \qquad \Omega \vdash \Xi_1 \triangleright t_1 : G_1[G_1'/Y] \leqslant \Xi_2 \triangleright t_2 : G_2[G_2'/Y] \qquad \exists Y.G_1 \leqslant \exists Y.G_2}{\Omega \vdash \Xi_1 \triangleright \mathsf{pack}\langle G_1', t_1\rangle \text{ as } \exists Y.G_1 : \exists Y.G_1 \leqslant \Xi_2 \triangleright \mathsf{pack}\langle G_2', t_2\rangle \text{ as } \exists Y.G_2 : \exists Y.G_2}$$

We are required to show

$$\Omega[\alpha/X] \vdash \Xi_1 \triangleright \mathsf{pack}\langle G_1'[\alpha/X], t_1[\hat{\alpha_1}/X]\rangle \text{ as } \exists Y.G_1[\alpha/X] :\leqslant \Xi_2 \triangleright :$$

$$\mathsf{pack}\langle G_2', t_2[\hat{\alpha_1}/X]\rangle \text{ as } \exists Y.G_2 : \exists Y.G_1[\alpha/X] \leqslant \exists Y.G_2[\alpha/X]$$

Note that $G_1'[\alpha/X] \leqslant G_2'[\alpha/X]$ by Proposition 10.26 and $\exists Y.G_1[\alpha/X] \leqslant \exists Y.G_2[\alpha/X]$ by Proposition 10.26. Therefore, we are required to prove $\Omega[\alpha/X] \vdash \Xi_1 \triangleright (t_1[\hat{\alpha_1}/X]) : G_1[G_1'/Y][\alpha/X] \leqslant \Xi_2 \triangleright (t_2[\hat{\alpha_2}/X]) : G_2[G_2'/Y][\alpha/X]$. But the results follows immediately by the induction hypothesis on $\Omega \vdash \Xi_1 \triangleright t_1 : G_1[G_1'/Y] \leqslant \Xi_2 \triangleright t_2 : G_2[G_2'/Y]$.

Case (unpack). We know that

$$(\leqslant\text{unpack}_\varepsilon)\dfrac{\Omega \vdash \Xi_1 \triangleright t_1 : \exists Y.G_1 \leqslant \Xi_2 \triangleright t_2 : \exists Y.G_2 \qquad \Omega, x : G_1 \sqsubseteq G_2 \vdash \Xi_1 \triangleright t_1' : G_1' \leqslant \Xi_2 \triangleright t_2' : G_2'}{\Omega \vdash \Xi_1 \triangleright \mathsf{unpack}\langle Y, x\rangle = t_1 \text{ in } t_1' : G_1' \leqslant \Xi_2 \triangleright \mathsf{unpack}\langle Y, x\rangle = t_2 \text{ in } t_2' : G_2'}$$

We are required to show

$$\Omega[\alpha/X] \vdash \Xi_1 \triangleright \mathsf{unpack}\langle Y, x\rangle = t_1[\hat{\alpha_1}/X] \text{ in } t_1'[\hat{\alpha_1}/X] : G_1'[\alpha/X] \leqslant \Xi_2 \triangleright \mathsf{unpack}\langle Y, x\rangle = t_2[\hat{\alpha_2}/X] \text{ in } t_2'[\hat{\alpha_2}/X] : G_2'[\alpha/X]$$

Therefore, we are required to prove $\Omega[\alpha/X] \vdash \Xi_1 \triangleright (t_1[\hat{\alpha_1}/X]) : \exists Y.G_1[\alpha/X] \leqslant \Xi_2 \triangleright (t_2[\hat{\alpha_2}/X]) : \exists Y.G_2[\alpha/X]$ and $\Omega[\alpha/X], x : G_1[\alpha/X] \sqsubseteq G_2[\alpha/X] \vdash \Xi_1 \triangleright (t_1'[\hat{\alpha_1}/X]) : G_1'[\alpha/X] \leqslant \Xi_2 \triangleright (t_2'[\hat{\alpha_2}/X]) : G_2'[\alpha/X]$. But the results follows immediately by the induction hypothesis on $\Omega \vdash \Xi_1 \triangleright t_1 : \exists Y.G_1 \leqslant \Xi_2 \triangleright t_2 : \exists Y.G_2$ and $\Omega, x : G_1 \sqsubseteq G_2 \vdash \Xi_1 \triangleright t_1' : G_1' \leqslant \Xi_2 \triangleright t_2' : G_2'$.

□

Proposition 10.22 (Lift Environment Precision). *If $G_1 \leqslant G_2$ and $\Xi_1 \leqslant \Xi_2$, then $\hat{G_1} \leqslant \hat{G_2}$, where $\hat{G_1} = lift_{\Xi_1}(G_1)$ and $\hat{G_2} = lift_{\Xi_2}(G_2)$.*

Proof. Remember that

$$lift_{\Xi}(G) = \begin{cases} lift_{\Xi}(G_1) \rightarrow lift_{\Xi}(G_2) & G = G_1 \rightarrow G_2 \\ \forall X.lift_{\Xi}(G_1) & G = \forall X.G_1 \\ \exists X.lift_{\Xi}(G_1) & G = \exists X.G_1 \\ lift_{\Xi}(G_1) \times lift_{\Xi}(G_2) & G = G_1 \times G_2 \\ \alpha^{lift_{\Xi}(\Xi(\alpha))} & G = \alpha \\ G & \text{otherwise} \end{cases}$$

The prove follows by the definition of $\hat{G}_1 = lift_{\Xi_1}(G_1)$ and induction on the structure of the type.

Case ($G_i = \exists X.G_i'$). We know that $G_1' \leqslant G_2'$. We are required to prove that $\exists X.lift_{\Xi_1}(G_1') \leqslant \exists X.lift_{\Xi_2}(G_2')$, or what is the same $lift_{\Xi_1}(G_1') \leqslant lift_{\Xi_2}(G_2')$. By the induction hypothesis on $G_1' \leqslant G_2'$ and $\Xi_1 \leqslant \Xi_2$ the result follows immediately.

□

Proposition 10.23. *If $G_1^* \leqslant G_2^*$ then $G_1^* \sqsubseteq G_2^*$.*

Proof. Examining $\leqslant$ rules.

Case ($\exists X.G_1 \leqslant \exists X.G_2$). We know that

$$\frac{G_1 \leqslant G_2}{\exists X.G_1 \leqslant \exists X.G_2}$$

By the induction hypothesis on $G_1 \leqslant G_2$, we know that $G_1 \sqsubseteq G_2$. We are required to prove that $\exists X.G_1 \sqsubseteq \exists X.G_2$, which follows immediately by the rule

$$\frac{G_1 \sqsubseteq G_2}{\exists X.G_1 \sqsubseteq \exists X.G_2}$$

□

Proposition 10.24. *If $G_1^* \sqsubseteq G_2^*$ and $G_1' \sqsubseteq G_2'$ then $G_1^*[G_1'/X] \sqsubseteq G_2^*[G_2'/X]$.*

Proof. Follow by induction on $G_1^* \sqsubseteq G_2^*$. We only take into account the existential type case.

Case ($\exists X.G_1 \sqsubseteq \exists X.G_2$). We know that

$$\frac{G_1 \sqsubseteq G_2}{\exists X.G_1 \sqsubseteq \exists X.G_2}$$

By the definition of $\sqsubseteq$, we know that $G_1 \sqsubseteq G_2$. We are required to prove that

$$(\exists X.G_1)[G_1'/X] = (\exists X.G_1[G_1'/X]) \sqsubseteq (\exists X.G_2[G_2'/X]) = (\exists X.G_2)[G_2'/X]$$

Or what is the same that $(G_1[G_1'/X]) \sqsubseteq (G_2[G_2'/X])$. But the result follows immediately by the induction hypothesis on $G_1 \sqsubseteq G_2$.

□

Proposition 10.25. *If $G_1 \sqsubseteq G_2$ and $G_1' \leqslant G_2'$ then $G_1[G_1'/X] \sqsubseteq G_2[G_2'/X]$.*

Proof. By Proposition 10.23 and Proposition 10.24 the results follows immediately. □

Proposition 10.26. *If $G_1 \leqslant G_2$ and $G_1' \leqslant G_2'$ then $G_1[G_1'/X] \leqslant G_2[G_2'/X]$.*

Proof. Straightforward induction on $G_1 \leqslant G_2$. Very similar to Proposition 10.24. □

Proposition 10.27. *If* $v_1 \leqslant t_2$ *then* $t_2 = v_2$.

Proof. Exploring $\leqslant$ rules. □

Proposition 10.28. *If* $\Xi_1 \vdash t_1 \leqslant \Xi_2 \vdash t_2$ *and* $\Xi_1 \triangleright t_1 \longmapsto \Xi_1' \triangleright t_1'$, *then* $\Xi_2 \triangleright t_2 \longmapsto \Xi_2' \triangleright t_2'$ *and* $\Xi_1' \vdash t_1' \leqslant \Xi_2' \vdash t_2'$.

Proof. If $\Xi_1 \vdash t_1 \leqslant \Xi_2 \vdash t_2$, we know that $\vdash t_1 \leqslant t_2 : G_1 \leqslant G_2$, $\Xi_1 \leqslant \Xi_2$, $\Xi_1 \vdash t_1 : G_1$ and $\Xi_2 \vdash t_2 : G_2$. We avoid the notation $\vdash t_1 \leqslant t_2 : G_1 \leqslant G_2$, and use $t_1 \leqslant t_2$ instead, for simplicity, when the typing environments are not relevant.

By induction on reduction $\Xi_1 \triangleright t_1 \longmapsto \Xi_1' \triangleright t_1'$. We only take into account the existential unpack case.

Case ($\Xi_1 \triangleright \mathsf{unpack}\langle X, x\rangle = t_{11}\ \mathsf{in}\ t_{12} \longmapsto \Xi_1' \triangleright \mathsf{unpack}\langle X, x\rangle = t_{11}'\ \mathsf{in}\ t_{12}$). By inspection of $\leqslant$, $t_2 = \mathsf{unpack}\langle X, x\rangle = t_{21}\ \mathsf{in}\ t_{22}$, where $t_{11} \leqslant t_{21}$ and $t_{12} \leqslant t_{22}$. By induction hypothesis on $\Xi_1 \triangleright t_{11} \longmapsto \Xi_1' \triangleright t_{11}'$, we know that $\Xi_2 \triangleright t_{21} \longmapsto \Xi_2' \triangleright t_{21}'$, where $\Xi_1' \vdash t_{11}' \leqslant \Xi_2' \vdash t_{21}'$. Then, by $\leqslant$, we know that $\Xi_1' \vdash \mathsf{unpack}\langle X, x\rangle = t_{11}'\ \mathsf{in}\ t_{12} \leqslant \Xi_2' \vdash \mathsf{unpack}\langle X, x\rangle = t_{21}'\ \mathsf{in}\ t_{22}$ and the result holds.

□