

1. Syntax
2. Consistency
3. Gradual typing rules
4. Intermediate language: typing + runtime
5. Cast insertion
6. Meta-theory

Source syntax

$$\begin{aligned}
 e &::= x \mid b \mid \lambda x:T.e \mid ee & \lambda x.e &\equiv \lambda x:?.e \\
 T &::= B \mid ? \mid T \rightarrow T \\
 v &::= b \mid \lambda x:T.e
 \end{aligned}$$

Consistency

$$\begin{aligned}
 T \sim T \quad T \sim ? \quad ? \sim T \\
 \frac{T_1 \sim T_1' \quad T_2 \sim T_2'}{T_1 \rightarrow T_2 \sim T_1' \rightarrow T_2'}
 \end{aligned}$$

Source typing

$$\begin{aligned}
 \frac{x:T \in \Gamma}{\Gamma \vdash x:T} (T\text{-Var}) & \quad \frac{\Gamma, x:T_1 \vdash e:T_2}{\Gamma \vdash \lambda x:T_1.e:T_1 \rightarrow T_2} (T\text{-Abs}) \\
 \frac{\Gamma \vdash e_1:T_1 \rightarrow T_2 \quad \Gamma \vdash e_2:T' \quad T' \sim T_1}{\Gamma \vdash e_1 e_2:T_2} (T\text{-App 1}) \\
 \frac{\Gamma \vdash e_1:?, \quad \Gamma \vdash e_2:T}{\Gamma \vdash e_1 e_2:?} (T\text{-App 2}) & \quad \frac{}{\Gamma \vdash b:B} (T\text{-B})
 \end{aligned}$$

Intermediate language

$e ::= \dots \mid \langle T_2 \Leftarrow T_1 \rangle e \mid \text{CastError}$

$v ::= \dots \mid \langle ? \Leftarrow T \rangle v \quad (T \neq ?)$

$E ::= E e \mid v E \mid \langle T_2 \Leftarrow T_1 \rangle E$ (evaluation contexts)

Typing:

$$\frac{\Gamma \vdash e : T_1}{\Gamma \vdash \langle T_2 \Leftarrow T_1 \rangle e : T_2}$$

$$\frac{}{\Gamma \vdash \text{CastError} : T}$$

Reduction:

$$\frac{T_1 \neq ?}{\langle T_2 \Leftarrow ? \rangle \langle ? \Leftarrow T_1 \rangle v \rightarrow \langle T_2 \Leftarrow T_1 \rangle v}$$

$$\frac{T_1 \not\sim T_2}{\langle T_2 \Leftarrow T_1 \rangle v \rightarrow \text{CastError}}$$

$$\frac{T_1 = T_2}{\langle T_2 \Leftarrow T_1 \rangle v \rightarrow v}$$

$$\frac{T_1 \neq T_2 \quad T_1 \sim T_2 \quad T_1 = T_{11} \rightarrow T_{12} \quad T_2 = T_{21} \rightarrow T_{22}}{\langle T_2 \Leftarrow T_1 \rangle v \rightarrow \lambda x : T_{21}. \langle T_{22} \Leftarrow T_{12} \rangle (v \langle T_{11} \Leftarrow T_{21} \rangle x)}$$

Cast insertion $\Gamma \vdash e \rightsquigarrow e' : T$

$$\frac{x : T \in \Gamma}{\Gamma \vdash x \rightsquigarrow x : T} \quad (\text{C-Var})$$

$$\frac{\Gamma, x : T_1 \vdash e \rightsquigarrow e' : T_2}{\Gamma \vdash \lambda x : T_1. e \rightsquigarrow \lambda x : T_1. e' : T_1 \rightarrow T_2} \quad (\text{C-Abs})$$

$$\frac{\Gamma \vdash e_1 \rightsquigarrow e'_1 : T_1 \rightarrow T_2 \quad \Gamma \vdash e_2 \rightsquigarrow e'_2 : T' \quad T' \sim T_1}{\Gamma \vdash e_1 e_2 \rightsquigarrow e'_1 \langle T_1 \Leftarrow T' \rangle e'_2 : T_2} \quad (\text{C-App1})$$

$$\Gamma \vdash e_1 e_2 \rightsquigarrow e'_1 \langle T_1 \Leftarrow T' \rangle e'_2 : T_2$$

$$\frac{}{\Gamma \vdash b \rightsquigarrow b : B} \quad (\text{C-B})$$

$$\frac{\Gamma \vdash e_1 \rightsquigarrow e'_1 : ? \quad \Gamma \vdash e_2 \rightsquigarrow e'_2 : T}{\Gamma \vdash e_1 e_2 \rightsquigarrow (\langle T \rightarrow ? \Leftarrow ? \rangle e'_1) e'_2 : ?} \quad (\text{C-App2})$$