Review: Proving progress

Let's quickly review the steps in the proof of the progress theorem:

- inversion lemma for typing relation
- canonical forms lemma
- progress theorem

Inversion

Lemma:

1. If $\Gamma \vdash \texttt{true} : \texttt{R}$, then $\texttt{R} = \texttt{Bool}$.
2. If $\Gamma \vdash \texttt{false} : \texttt{R}$, then $\texttt{R} = \texttt{Bool}$.
3. If $\Gamma \vdash \texttt{if } \texttt{t}_1 \texttt{ then } \texttt{t}_2 \texttt{ else } \texttt{t}_3 : \texttt{R}$, then $\Gamma \vdash \texttt{t}_1 : \texttt{Bool}$ and $\Gamma \vdash \texttt{t}_2, \texttt{t}_3 : \texttt{R}$.
4. If $\Gamma \vdash \texttt{x} : \texttt{R}$, then

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4. If $\Gamma \vdash \texttt{x} : \texttt{R}$, then $\texttt{x:R} \in \Gamma$.
5. If $\Gamma \vdash \lambda\texttt{x:T}_1\texttt{.t}_2 : \texttt{R}$, then

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4. If $\Gamma \vdash \mathtt{x} : \mathtt{R}$, then $\mathtt{x}{:}\mathtt{R} \in \Gamma$.
5. If $\Gamma \vdash \lambda \mathtt{x}{:}\mathtt{T}_1.\mathtt{t}_2 : \mathtt{R}$, then $\mathtt{R} = \mathtt{T}_1 {\to} \mathtt{R}_2$ for some $\mathtt{R}_2$ with $\Gamma, \mathtt{x}{:}\mathtt{T}_1 \vdash \mathtt{t}_2 : \mathtt{R}_2$.
6. If $\Gamma \vdash \mathtt{t}_1\ \mathtt{t}_2 : \mathtt{R}$, then

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6. If $\Gamma \vdash \mathtt{t}_1\ \mathtt{t}_2 : \mathtt{R}$, then there is some type $\mathtt{T}_{11}$ such that $\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_{11} \rightarrow \mathtt{R}$ and $\Gamma \vdash \mathtt{t}_2 : \mathtt{T}_{11}$.

Canonical Forms

Lemma:

Canonical Forms

Lemma:

1. If $\mathtt{v}$ is a value of type $\mathtt{Bool}$, then $\mathtt{v}$ is either $\mathtt{true}$ or $\mathtt{false}$.
2. If $\mathtt{v}$ is a value of type $\mathtt{T_1} \to \mathtt{T_2}$, then $\mathtt{v}$ has the form $\lambda \mathtt{x}:\mathtt{T_1}.\mathtt{t_2}$.

Progress

Theorem: Suppose $\mathtt{t}$ is a closed, well-typed term (that is, $\vdash \mathtt{t} : \mathtt{T}$ for some $\mathtt{T}$). Then either $\mathtt{t}$ is a value or else there is some $\mathtt{t}'$ with $\mathtt{t} \longrightarrow \mathtt{t}'$.

Preservation (and Weaking, Permutation, Substitution)

Preservation

Theorem: If $\Gamma \vdash \mathtt{t} : \mathtt{T}$ and $\mathtt{t} \longrightarrow \mathtt{t}'$, then $\Gamma \vdash \mathtt{t}' : \mathtt{T}$.

Steps of proof:

- Weakening
- Permutation
- Substitution preserves types
- Reduction preserves types (i.e., preservation)

Weakening and Permutation

Weakening tells us that we can *add assumptions* to the context without losing any true typing statements.

Lemma: If $\Gamma \vdash \mathtt{t} : \mathtt{T}$ and $\mathtt{x} \notin \mathit{dom}(\Gamma)$, then $\Gamma, \mathtt{x}{:}\mathtt{S} \vdash \mathtt{t} : \mathtt{T}$.

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Permutation tells us that the order of assumptions in (the list) Γ does not matter.

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Theorem: If $\Gamma \vdash \mathtt{t} : \mathtt{T}$ and $\mathtt{t} \longrightarrow \mathtt{t}'$, then $\Gamma \vdash \mathtt{t}' : \mathtt{T}$.

Proof: By induction

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Proof: By induction on typing derivations.

Which case is the hard one??

Preservation

Theorem: If $\Gamma \vdash \mathtt{t} : \mathtt{T}$ and $\mathtt{t} \longrightarrow \mathtt{t}'$, then $\Gamma \vdash \mathtt{t}' : \mathtt{T}$.

Proof: By induction on typing derivations.

Case T-APP: Given $\mathtt{t} = \mathtt{t}_1\ \mathtt{t}_2$

$\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_{11} \to \mathtt{T}_{12}$

$\Gamma \vdash \mathtt{t}_2 : \mathtt{T}_{11}$

$\mathtt{T} = \mathtt{T}_{12}$

Show $\Gamma \vdash \mathtt{t}' : \mathtt{T}_{12}$

Preservation

Theorem: If $\Gamma \vdash \mathtt{t} : \mathtt{T}$ and $\mathtt{t} \longrightarrow \mathtt{t}'$, then $\Gamma \vdash \mathtt{t}' : \mathtt{T}$.

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Subcase: $\mathtt{t}_1 = \lambda \mathtt{x}{:}\mathtt{T}_{11}.\ \mathtt{t}_{12}$

$\mathtt{t}_2$ a value $\mathtt{v}_2$

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Proof: By induction on typing derivations.

Case T-APP: Given

$$\begin{aligned} &\mathtt{t} = \mathtt{t}_1\ \mathtt{t}_2 \\ &\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_{11} \rightarrow \mathtt{T}_{12} \\ &\Gamma \vdash \mathtt{t}_2 : \mathtt{T}_{11} \\ &\mathtt{T} = \mathtt{T}_{12} \end{aligned}$$

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Uh oh.

Preservation

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Proof: By induction on typing derivations.

Case T-App: Given $\mathtt{t} = \mathtt{t}_1\ \mathtt{t}_2$

$\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_{11} \rightarrow \mathtt{T}_{12}$

$\Gamma \vdash \mathtt{t}_2 : \mathtt{T}_{11}$

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Uh oh. What do we need to know to make this case go through??

The "Substitution Lemma"

Lemma: If $\Gamma, \mathtt{x}{:}\mathtt{S} \vdash \mathtt{t} : \mathtt{T}$ and $\Gamma \vdash \mathtt{s} : \mathtt{S}$, then $\Gamma \vdash [\mathtt{x} \mapsto \mathtt{s}]\mathtt{t} : \mathtt{T}$.

I.e., "Types are preserved under substitition."

The "Substitution Lemma"

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Proof: By induction on the *depth* of a derivation of $\Gamma, \mathtt{x}{:}\mathtt{S} \vdash \mathtt{t} : \mathtt{T}$. Proceed by cases on the final typing rule used in the derivation.

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Proof: By induction on the *depth* of a derivation of $\Gamma, \mathtt{x}{:}\mathtt{S} \vdash \mathtt{t} : \mathtt{T}$. Proceed by cases on the final typing rule used in the derivation.

Case T-App:

$$\begin{array}{l}
\mathtt{t} = \mathtt{t}_1\ \mathtt{t}_2 \\
\Gamma, \mathtt{x}{:}\mathtt{S} \vdash \mathtt{t}_1 : \mathtt{T}_2 {\to} \mathtt{T}_1 \\
\Gamma, \mathtt{x}{:}\mathtt{S} \vdash \mathtt{t}_2 : \mathtt{T}_2 \\
\mathtt{T} = \mathtt{T}_1
\end{array}$$

By the induction hypothesis, $\Gamma \vdash [\mathtt{x} \mapsto \mathtt{s}]\mathtt{t}_1 : \mathtt{T}_2 {\to} \mathtt{T}_1$ and $\Gamma \vdash [\mathtt{x} \mapsto \mathtt{s}]\mathtt{t}_2 : \mathtt{T}_2$. By T-App, $\Gamma \vdash [\mathtt{x} \mapsto \mathtt{s}]\mathtt{t}_1\ [\mathtt{x} \mapsto \mathtt{s}]\mathtt{t}_2 : \mathtt{T}$, i.e., $\Gamma \vdash [\mathtt{x} \mapsto \mathtt{s}](\mathtt{t}_1\ \mathtt{t}_2) : \mathtt{T}$.

The "Substitution Lemma"

Lemma: If $\Gamma, \mathtt{x}{:}\mathtt{S} \vdash \mathtt{t} : \mathtt{T}$ and $\Gamma \vdash \mathtt{s} : \mathtt{S}$, then $\Gamma \vdash [\mathtt{x} \mapsto \mathtt{s}]\mathtt{t} : \mathtt{T}$.

Proof: By induction on the *depth* of a derivation of $\Gamma, \mathtt{x}{:}\mathtt{S} \vdash \mathtt{t} : \mathtt{T}$. Proceed by cases on the final typing rule used in the derivation.

Case T-VAR: $\mathtt{t} = \mathtt{z}$
with $\mathtt{z}{:}\mathtt{T} \in (\Gamma, \mathtt{x}{:}\mathtt{S})$

There are two sub-cases to consider, depending on whether $\mathtt{z}$ is $\mathtt{x}$ or another variable. If $\mathtt{z} = \mathtt{x}$, then $[\mathtt{x} \mapsto \mathtt{s}]\mathtt{z} = \mathtt{s}$. The required result is then $\Gamma \vdash \mathtt{s} : \mathtt{S}$, which is among the assumptions of the lemma. Otherwise, $[\mathtt{x} \mapsto \mathtt{s}]\mathtt{z} = \mathtt{z}$, and the desired result is immediate.

The "Substitution Lemma"

Lemma: If $\Gamma, \mathtt{x}{:}\mathtt{S} \vdash \mathtt{t} : \mathtt{T}$ and $\Gamma \vdash \mathtt{s} : \mathtt{S}$, then $\Gamma \vdash [\mathtt{x} \mapsto \mathtt{s}]\mathtt{t} : \mathtt{T}$.

Proof: By induction on the *depth* of a derivation of $\Gamma, \mathtt{x}{:}\mathtt{S} \vdash \mathtt{t} : \mathtt{T}$. Proceed by cases on the final typing rule used in the derivation.

Case T-ABS: $\mathtt{t} = \lambda \mathtt{y}{:}\mathtt{T}_2.\mathtt{t}_1 \qquad \mathtt{T} = \mathtt{T}_2 {\to} \mathtt{T}_1$

$\Gamma, \mathtt{x}{:}\mathtt{S}, \mathtt{y}{:}\mathtt{T}_2 \vdash \mathtt{t}_1 : \mathtt{T}_1$

By our conventions on choice of bound variable names, we may assume $\mathtt{x} \neq \mathtt{y}$ and $\mathtt{y} \notin FV(\mathtt{s})$. Using *permutation* on the given subderivation, we obtain $\Gamma, \mathtt{y}{:}\mathtt{T}_2, \mathtt{x}{:}\mathtt{S} \vdash \mathtt{t}_1 : \mathtt{T}_1$. Using *weakening* on the other given derivation ($\Gamma \vdash \mathtt{s} : \mathtt{S}$), we obtain $\Gamma, \mathtt{y}{:}\mathtt{T}_2 \vdash \mathtt{s} : \mathtt{S}$. Now, by the induction hypothesis, $\Gamma, \mathtt{y}{:}\mathtt{T}_2 \vdash [\mathtt{x} \mapsto \mathtt{s}]\mathtt{t}_1 : \mathtt{T}_1$. By T-ABS, $\Gamma \vdash \lambda \mathtt{y}{:}\mathtt{T}_2.\ [\mathtt{x} \mapsto \mathtt{s}]\mathtt{t}_1 : \mathtt{T}_2 {\to} \mathtt{T}_1$, i.e. (by the definition of substitution), $\Gamma \vdash [\mathtt{x} \mapsto \mathtt{s}]\lambda \mathtt{y}{:}\mathtt{T}_2.\ \mathtt{t}_1 : \mathtt{T}_2 {\to} \mathtt{T}_1$.